Axionic Wormholes :

More on their Classical and Quantum Aspects

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Abstract

As a system which is known to admit classical wormhole instanton solutions, Einstein-Kalb-Ramond (KR) antisymmetric tensor theory is revisited. As an untouched issue, the existence of fermionic zero modes in the background of classical axionic wormhole spacetime and its physical implications is addressed. In particular, in the context of a minisuperspace quantum cosmology model based on this Einstein-KR antisymmetric tensor theory, "quantum wormhole", defined as a state represented by a solution to the Wheeler-DeWitt equation satisfying an appropriate wormhole boundary condition, is discussed. An exact, analytic wave function for quantum wormholes is actually found. Finally, it is proposed that the minisuperspace model based on this theory in the presence of the cosmological constant may serve as an interesting simple system displaying an overall picture of entire universe's history from the deep quantum domain all the way to the classical domain.

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I. Introduction

At present, in the absence of a complete, consistent theory of quantum gravity, a systematic formulation of the laws of physics around the Planck scale seems beyond our scope. Nevertheless, one may still wish to learn something about these laws by studying possible predictions from conventional approaches toward the construction of quantum gravity such as the canonical quantization of general relativity or at least from its semiclassical approximations. Indeed, some time ago, such attempts and the associated debates had been circulating in the theoretical physics society which may be summarized and called as "effects of topology change in spacetime on low energy physics" [1-3]. Known by a more popular name, "Euclidean wormhole physics" [1-6], these attempts can be recalled as follows. As pointed out first by Wheeler, if one identifies the spacetime metric as the relevant gravitational field subject to the quantization, the topology of spacetime is expected to fluctuate as well on scales of the order of the Planck length $l_p = M_p^{-1}$. And of all types of conceivable spacetime fluctuations, our major concern is the "wormhole configuration" which is an object that can be loosely defined as the instanton which is a saddle point of the Euclidean action making dominant contribution to the topology changing transition amplitude. Then one of the most crucial effects the wormhole (or more generally, these spacetime fluctuations) may have on low-energy physics could be the possible effective loss of quantum coherence [1-3]. For example, one may speculate the situation where "baby universes" are pinched off and carry away information. Then this kind of stereotypical information loss can lead to an effective loss of quantum coherence as viewed by the macroscopic observer who cannot measure the quantum state of the baby universes. At this point, it may be interesting to recall other types of loss of quantum coherence in semiclassical quantum gravity known thus far. Namely, from the study of dynamical quantum fields in the background of black hole spacetimes, Hawking [9] discovered the evaporation of black holes via emitting quanta. He then argued that this quantum black hole radiation necessarily leads to the "evolution of pure states into mixed states" [10] which obviously signals just another type of information loss down to black holes. Although the calculation involved in the demonstration of this

quantum black hole radiation has been carried out in the context of semiclassical quantum gravity, the associated quantum incoherence seems generic and thus may well survive even the full quantum gravitational treatment. Therefore these two types of information loses in the wormhole and in the black hole physics lead us to suspect that dynamics at Planck scale result in the loss of quantum coherence on general grounds. In a more careful and concrete analysis, however, these arguments should be taken with some caution. For example, there is an argument by Coleman [3] that in the context of "many universe" interpretation (i.e., the third quantization formalism), the quantum incoherence will not be observed, namely the quantum coherence will be restored. And there is another argument by Giddings and Strominger [4] which states that, although at first glance the quantum incoherence is expected to violate all the conservation laws, what really happens may be that probably currents associated with local symmetries are exactly conserved while those associated only with global symmetries are not. Of all the possible effects of the fluctuations in spacetime topology on the low energy physics, the most provocative one that immediately attracted enormous excitement was the advocation initiated by Baum [1] and by Hawking [1] and then refined later by Coleman [3] that the wormholes have ultimately an effect of turning all the constants of nature into "dynamical random variables". Thus this Baum-Hawking-Coleman (BHC) mechanism leads to a striking conclusion that the effects of (particularly) wormholes introduce into the low energy physics a fundamental quantum indeterminancy of the values of the constants of nature which can be thought of as an additional degree of uncertainty over the usual uncertainty in quantum mechanics. Unlike the issue of the loss of quantum coherence discussed earlier, however, this BHC-mechanism does not simply imply the elimination of the classical predictability of nature by the effect of quantum gravity. For instance, the BHC-mechanism actually leads to the prediction that the most probable value of the fully-renormalized cosmological constant is zero, indeed in exact agreement with the observation. For the detailed arguments involved in the BHC-mechanism particularly concerning the most probable value of the fully-renormalized cosmological constant, we refer the reader to the literature [1,3]. But it seems fair to mention that the formulation of wormhole physics

particularly the one put forward by Coleman is not without some inherent flaws. First, logically the Coleman's wormhole physics formulation had faced some severe criticisms such as "sliding of Newton's constant problem" [7] and "large wormhole catastrophe" [8] which seem to be unfortunately quite generic. However, it seems that the most fundamental and crucial difficulty associated with Coleman's formulation is the use of saddle point approximation to the Euclidean path integral for quantum gravity since, as is well-known, the Euclidean Einstein-Hilbert action is not bounded below [11].

Now, then, note that both issues discussed thus far, namely the loss of quantum coherence and the determination of probability distribution for constants of nature (which are now random variables), are clearly based on the assumption that there really are wormhole instantons as saddle points of the Euclidean action of the theory under consideration. Therefore unless one can demonstrate that there are large class of theories comprised of gravity with or without matter which admit Planck-sized wormhole instantons as solutions to the classical field equations, the discussion above on interesting effects of wormholes on low energy physics will lose much of its meaning. Unfortunately, thus far only a handful of restricted classes of theories are known to possess classical wormhole instanton solutions and they include ; Einstein-Kalb-Ramond (KR) antisymmetric tensor theory [4], Einstein-Yang-Mills theory [5] and Einstein-complex scalar field theory in the presence of spontaneous symmetry breaking [6]. The classical wormhole instanton solutions and the semiclassical analysis of their effects on low energy physics in these theories had been thoroughly studied in the literature. Here in the present work, we revisit the Einstein-KR antisymmetric tensor theory of Giddings and Strominger [4] which is a "classic" system known to admit classical, Euclidean wormhole instanton solution. And as was emphasized by Giddings and Strominger [4], the peculiar feature that renders this theory to possess an Euclidean wormhole solution is the wrong sign in the Euclidean energy-momentum tensor when the antisymmetric tensor field strength is represented by an axion field via duality transformation. One may wonder why anybody should repeatedly go through a well-studied theory like this one. Although the "classical" wormhole instanton as a solution to the classical field

equations and some of its effects on low energy physics had been studied extensively, almost no attempt has been made concerning the serious study of "quantum" wormholes in the same theory. Besides some important aspects of the classical wormhole physics such as the existence of fermion zero modes in the background of classical axionic wormhole spacetime and its physical implications have not been addressed. It is these kinds of untouched but interesting issues that the present work attempts to deal with. Firstly, the investigation of the existence of the fermion zero modes and their physical implications have been carried out by Hosoya and Ogura[5] in wormhole physics in Einstein-Yang-Mills theory. Thus we follow similar avenue to the one taken there to study the physics associated with the fermion zero modes in our Einstein-KR antisymmetric tensor theory. And to do so, we need to introduce interactions between the KR antisymmetric tensor field and the fermion field possessing both the general covariance and the local gauge-invariance which has never been considered thus far. (Note that this interaction should be distinguished from the derivatively coupled pseudoscalar Goldstone boson (axion) field - fermion field interactions in effective field theories.) Secondly, we describe briefly the approach we shall employ to explore the physics of quantum wormholes in our theory. And to do so, it seems necessary to distinguish between the definition of "classical" wormholes and that of "quantum" wormholes. In the classical sense, wormholes are Euclidean metrics which are solutions to the Euclidean classical field equations representing spacetimes consisting of two asymptotically Euclidean regions joined by a narrow tube or throat. In the quantum regime, on the other hand, and particularly in the context of the canonical quantum cosmology, quantum wormholes may be identified with a state or an excitation represented by a solution to the Wheeler-DeWitt equation satisfying a certain boundary condition describing the wormhole configuration. An widely-accepted such "wormhole boundary condition" is the one advocated by Hawking and Page [12]. And it states that wormhole wave functions are supposed to behave in such a way that they are damped, say, exponentially for large 3-geometries ($\sqrt{h} \to \infty$) and are regular in some suitable way when the 3-geometry collapses to zero ($\sqrt{h} \to 0$). Thus we shall construct a minisuperspace quantum cosmology model possessing SO(4)-symmetry based

on the Einstein-KR antisymmetry tensor field and attempt to solve associated Wheeler-DeWitt equation. As we shall see later on, we find an exact, analytic solution satisfying the "wormhole boundary condition" stated above and identify it with a wormhole wave function, namely a universe wave function for quantum wormholes.

This paper is organized as follows : In sect.2, we recapitulate classical wormhole instanton solutions and the semiclassical analysis of their effects on low energy physics in Einstein-KR antisymmetric tensor theory. In sect.3, we address the issue concerning the fermion zero modes in the background of classical axionic wormhole spacetime and their physical implications. Sect.4 will be devoted to the study of quantum wormholes in this theory employing the approach described above. Finally in sect.5, we summarize the results of our study and discuss their physical implications.

II. Axionic Wormhole Instantons Revisited

We begin by reviewing the classical Einstein-KR antisymmetric tensor theory.

Consider a system comprised of an axion (described by a rank-three antisymmetric tensor field strength $H_{\mu\nu\lambda}$) coupled to gravity. This Einstein-KR antisymmetric tensor (EAT) theory is represented by the Euclidean action [4]

$$I_{EAT} = \int_M d^4x\sqrt{g}[-\frac{M_p^2}{16\pi}R + f_a^2 H_{\mu\nu\lambda}H^{\mu\nu\lambda}] - \int_{\partial M} d^3x\sqrt{h}\frac{M_p^2}{8\pi}(K - K_0) \tag{1}$$

where we added Gibbons-Hawking gravitational boundary term on ∂M with h being the metric induced on ∂M and K being the trace of the second fundamental form of ∂M. Here f_a is the Peccei-Quinn scale and $H = dB$ is the field strength tensor of the antisymmetric tensor gauge field $B_{\mu\nu}$ of Kalb-Ramond

$$H = dB. \tag{2}$$

Then since $H = dB$, we have the Bianchi identity $dH = 0$, namely

$$\nabla_{[\rho}H_{\mu\nu\lambda]} = 0. \tag{3}$$

In addition, $H(H_{\mu\nu\lambda})$ is invariant under the gauge transformation (since $d^2 \equiv 0$)

$$B \rightarrow B + d\Lambda \tag{4}$$

$$\text{or} \qquad \mathrm{B}_{\mu\nu} \rightarrow B_{\mu\nu} + (\partial_\mu \Lambda_\nu - \partial_\nu \Lambda_\mu).$$

Now by extremizing the action above with respect to the metric $g_{\mu\nu}$ and the Kalb-Ramond antysymmetric tensor field $B_{\mu\nu}$, one gets the classical field equations

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{8\pi}{M_p^2} f_a^2 T_{\mu\nu} \tag{5}$$

$$\text{with} \quad T_{\mu\nu} = 6(H_{\mu\alpha\beta} H_\nu^{\alpha\beta} - \frac{1}{6} g_{\mu\nu} H_{\alpha\beta\gamma} H^{\alpha\beta\gamma}),$$

$$d^* H = 0 \qquad \text{or} \qquad \nabla_\mu H^{\mu\nu\lambda} = 0, \tag{6}$$

$$dH = 0 \qquad \text{or} \qquad \nabla_{[\rho} H_{\mu\nu\lambda]} = 0 \tag{7}$$

where we included the Bianchi identity in the last line.

Now from the classical field equation for the Kalb-Ramond field $d^* H = 0$, we now can define the "conserved axion current"

$$j = {}^* H \tag{8}$$

since $dj = d^* H = 0$. Further, we can write, at least, locally $j = dA$ (since $d^2 \equiv 0$) with $A(x)$ denoting the "axion field"

$$H = {}^*(dA)$$

$$\text{or} \quad H_{\mu\nu\lambda} = \epsilon_{\mu\nu\lambda\beta} (\partial^\beta A) \tag{9}$$

only on-shell. Here, caution must be exercised: if the manifold M is not simply connected, the pseudoscalar $A(x)$ may not be globally defined. Now, on-shell, the energy-momentum tensor for the Kalb-Ramond field can be expressed in terms of the axion field (at least, locally)

$$T^{\mu\nu} = \frac{2}{\sqrt{g}} \frac{\delta I_{EAT}}{\delta g_{\mu\nu}},$$

$$\text{namely,} \quad T_{\mu\nu}(A) = -12[(\nabla_\mu A)(\nabla_\nu A) - \frac{1}{2} g_{\mu\nu} (\nabla_\lambda A)(\nabla^\lambda A)]. \tag{10}$$

As was first pointed out by Giddings and Strominger [4], the energy-momentum tensor expressed in terms of the axion field $A(x)$ has wrong sign when compared with that of ordinary, minimally-coupled scalar field. Thus, the Euclidean behavior of the axion field (associated with the KR antisymmetric tensor field) coupled to gravity is radically different from that of an ordinary scalar field and this is essentially responsible for the fact that wormhole instantons do exist in this Einstein-KR antisymmetric tensor theory.

Now, we look for a Euclidean $SO(4)$-symmetric wormhole solution which is an instanton describing the nucleation of a Planck-sized baby (spatially-closed; $k = +1$) FRW universe. To this end, we begin by taking $SO(4)$-symmetric ansắtz for the Euclidean metric and Kalb-Ramond antisymmetric tensor field as

$$\begin{aligned} ds^2 &= g_{\mu\nu}dx^\mu dx^\nu = \delta_{AB}e^A \otimes e^B \\ &= N^2(\tau)d\tau^2 + a^2(\tau)d\Omega_3^2, \qquad (11) \\ H &= h(\tau)\epsilon \qquad \text{or} \qquad H_{\mu\nu\lambda} = h(\tau)\epsilon_{\mu\nu\lambda} \qquad (\mu,\nu,\lambda \neq \tau) \end{aligned}$$

where e^A $(A, B = 0, 1, 2, 3)$ are non-coordinate basis 1-forms

$$e^A = \{e^0 = Nd\tau, \quad e^a = a\sigma^a\}, \qquad (12)$$

$N(\tau)$ and $a(\tau)$ are lapse function and scale factor respectively and $d\Omega_3^2 = \sigma^a \otimes \sigma^a$ denotes the line element on 3-sphere S^3 with $\{\sigma^a\}$ $(a = 1, 2, 3)$ forming a basis on the S^3 and $\epsilon = \frac{1}{3!}\epsilon_{\mu\nu\lambda}dx^\mu \wedge dx^\nu \wedge dx^\lambda = \sqrt{h}d^3x$ is the volume 3-form normalized so that

$$\int_{S^3} \epsilon = \int_{S^3} d^3x\sqrt{h} = 2\pi^2 a^3(\tau). \qquad (13)$$

Then for later use, we note that in terms of this SO(4)-symmetric ansắtz, the energy-momentum tensor for the KR antisymmetric tensor field becomes

$$\begin{aligned} T_{\mu\nu} &= 6(H_{\mu\alpha\beta}H_\nu^{\alpha\beta} - \frac{1}{6}g_{\mu\nu}H_{\alpha\beta\gamma}H^{\alpha\beta\gamma}) \\ &= 6h^2(\tau)(2h_{\mu\nu} - g_{\mu\nu}) \qquad (14) \\ &\text{with} \qquad h_{\tau\tau} = h_{\mu\tau} = h_{\tau\nu} = 0. \end{aligned}$$

We now have to solve the "coupled" Einstein-antisymmetric tensor field equations. Fortunately, however, the antisymmetric tensor sector of field equations, namely the Euler-Lagrange's equation of motion and the Bianchi identity are satisfied, $d^*H = 0 = dH$, if we set [4]

$$h(\tau) = \frac{n}{f_a^2 a^3(\tau)} \tag{15}$$

so that

$$\int_{S^3} H = \int_{S^3} h(\tau)\epsilon = \frac{2\pi^2 n}{f_a^2} \tag{16}$$

where f_a is, as introduced, the Peccei-Quinn scale and n is a free parameter which, in string theory, is quantized. In fact, once we set $H_{\mu\nu\lambda} = h(\tau)\epsilon_{\mu\nu\lambda}$, it automatically satisfies the Euler-Lagrange's equation of motion $\nabla_\mu H^{\mu\nu\lambda} = 0$. Note here that in contrast to the original formulation by Giddings and Strominger [4] where they obtained this SO(4)-symmetric expression for the KR antisymmetric tensor field strength as a solution to the classical field equation and the Bianchi identity, here we stress that it can be "derived" simply from the definition $H = dB$ and the Bianchi identity $dH = 0$ *without* imposing on-shell condition. To see this briefly, using the SO(4)-symmetric ansǟtz $H_{\mu\nu\lambda} = h(\tau)\epsilon_{\mu\nu\lambda}$, start with

$$\begin{aligned} H &= \frac{1}{3!} H_{\mu\nu\lambda} dx^\mu \wedge dx^\nu \wedge dx^\lambda \\ &= \frac{1}{3!} h(\tau)\epsilon_{abc}(e^a \wedge e^b \wedge e^c) \end{aligned}$$

where $a, b, c = 1, 2, 3$ since $\mu, \nu, \lambda \neq \tau$. Then consider

$$dH = \frac{1}{3!}[h'(\tau)\epsilon_{abc}(d\tau \wedge e^a \wedge e^b \wedge e^c) + h(\tau)\epsilon_{abc} d(e^a \wedge e^b \wedge e^c)].$$

Since $\{\sigma^a\}$, which form a basis on the three-sphere S^3, satisfy the SU(2) "Maurer-Cartan" structure equation $d\sigma^a = \frac{1}{2}\epsilon^{abc}\sigma^b \wedge \sigma^c$, we have, using eq.(12),

$$dH = \frac{1}{3!}[(\frac{h'}{N} + 3\frac{h}{N}\frac{a'}{a})\epsilon_{abc}(e^0 \wedge e^a \wedge e^b \wedge e^c)].$$

Finally, imposing the Bianchi identity $dH = 0$ (since $H = dB$) yields

$$h' + 3(\frac{a'}{a})h = 0$$

of which the solution takes the form given in eq.(15),

$$h(\tau) = \frac{n}{f_a^2 a^3(\tau)}.$$

Therefore, this SO(4)-symmetric ansätz for the KR antisymmetric field strength in eq.(15) remains perfectly valid even off-shell as well as on-shell. In other words, this expression for $H_{\mu\nu\lambda}$ can be used for both classical and quantum treatments that we shall discuss later on in the section of quantum axionic wormholes.

Now what remains is to solve the Einstein field equations which are no longer coupled equations and reduce to non-linear equations of the scale factor $a(\tau)$ alone. Besides, we only need to consider the time-time component of the Einstein equations since the rest of the equations are implied by the Bianchi identity (i.e., energy-momentum conservation). Thus from eqs. (5), (11) and (14), consider the $\tau\tau$-component of the Einstein equations

$$\frac{3M_p^2}{16\pi}[(\frac{a'}{a})^2 - \frac{N^2}{a^2}] = -\frac{3n^2}{f_a^2}\frac{N^2}{a^6}.$$

where the "prime" denotes the derivative with respect to the Euclidean time τ.

(1) With the gauge-fixing $N(\tau) = 1$:

$$(\frac{a'}{a})^2 - \frac{1}{a^2} = -\frac{r^4}{a^6} \qquad \left(r^2 = \frac{4\sqrt{\pi}n}{M_p f_a} = a^2(\tau = 0)\right). \tag{17}$$

Since we set r^2 as the value of a^2 at $\tau = 0$, i.e., $a^2(\tau = 0) = r^2$, it can be integrated to yield

$$\begin{aligned} \tau = \int_0^\tau d\tau' &= \int_{a(0)=r^2}^{a(\tau)} \frac{a^2 da}{\sqrt{(a^2 + r^2)(a^2 - r^2)}} \\ &= \frac{r}{\sqrt{2}}F[cos^{-1}(\frac{r}{a}), \frac{1}{\sqrt{2}}] - \sqrt{2}rE[cos^{-1}(\frac{r}{a}), \frac{1}{\sqrt{2}}] + \frac{1}{a}\sqrt{a^4 - r^4} \end{aligned} \tag{18}$$

where F and E are elliptic integrals of the first and second kinds, respectively. This Euclidean wormhole instanton solution is characterized by one free parameter, n, which, as mentioned, is quantized in string theory. Note the "asymptotic behavior" of this Euclidean wormhole solution

$$a(\tau) \to \tau, \qquad \text{as} \quad \tau \to \pm\infty.$$

The wormhole instanton solution we obtained is drawn in Fig. 1. Since $a^2(\tau) \to \tau^2$ as $\tau \to \pm\infty$, there are two asymptotically Euclidean regions. They are joined by a "throat" whose cross sections are S^3's. The axion current ${}^*j$ has total integrated flux n/f_a^2 through the throat. As it stands, it is difficult to ascribe a physical interpretation to this instanton configuration because of the two asymptotic regions. (However, it might represent communication between two different universes). The situation can be improved by slicing the wormhole instanton in half through the minimal surface of the throat which is drawn in Fig. 2. It represents tunnelling from an initial hypersurface Σ_i with topology R^3 to a final hypersurface Σ_f with topology $R^3 \oplus S^3$. It describes nucleation of a baby (spatially-closed) FRW-universe created at its moment of time symmetry. Note that, in order for this instanton solution in Fig. 2 to describe a reasonable tunnelling process, it is crucial that the fields involved (i.e., the metric field $a(\tau)$ and the Kalb-Ramond field strength $H_{\mu\nu\lambda} = h(\tau)\epsilon_{\mu\nu\lambda}$ $(\mu, \nu, \lambda \neq \tau)$) should take appropriate values. Namely, it must be true that the fields and their first time derivatives on Σ_i and Σ_f are all real when analytically continued back to the Lorentzian spacetime. This is obvious for the R^3 part of Σ_i and Σ_f. On the S^3 portion, the time derivative of the metric vanishes because it is a minimal surface. The time components of $H_{\mu\nu\lambda} = h(\tau)\epsilon_{\mu\nu\lambda}$ vanish because it is a 3-form tangent to the spacelike hypersurface. Thus the instanton obtained above does obey exactly the right boundary conditions for the description of the tunnelling $R^3 \to R^3 \oplus S^3$. An additional important feature which characterizes this instanton is the axion current through the throat of the wormhole $(R \otimes S^3)$ and the axion charge on the non-contractable 3-spheres (S^3). Axion current through the wormhole throat is from eq.(8),

$$^*j = H =^* (dA) \tag{19}$$

which is a 3-form and is conserved owing to Bianchi identity $d^*j = dH = 0$. The axion current flux through the wormhole throat is

$$\int_{S^3} {}^*j = \int_{S^3} H = \int_{S^3} h(\tau)\epsilon = \int_{S^3} \epsilon \frac{n}{f_a^2 a^3(\tau)} = 2\pi^2 (\frac{n}{f_a^2}). \tag{20}$$

Axion charge on the cross section of the wormhole throat is

$$q = f_a^2 \int_{S^3(\tau=0)} {}^*j = f_a^2 \int_{S^3(\tau=0)} H = 2\pi^2 n. \tag{21}$$

(In string theory, global anomalies in the string sigma model lead to quantization of n in this axion charge.) The observer on R^3 will measure a change of $\frac{1}{2\pi^2}\triangle q = (-n)$ in the axion charge, since the baby universe pinches off n-units of axion charge. This, of course, would be rather puzzling to the observer on R^3, who cannot observe the charge on the baby universe, since he or she may believe that axion charge is conserved due to the (unbroken since $f_a < M_p$) Peccei-Quinn symmetry of the action. This effective charge non-conservation can be understood as a result of the breakdown of quantum coherence due to information loss to baby universes (in the similar spirit to information loss in black hole evaporatioin by Hawking effect). Next, we evaluate the (wormhole) instanton actioin I_{EAT}(instanton), namely the minimum Euclidean action of the instanton configuration which makes dominant contribution to the tunnelling amplitude. Namely, into the Euclidean action of this Einstein-antisymmetric tensor theory given earlier

$$I_{EAT} = \int_M d^4x\sqrt{g}[-\frac{M_p^2}{16\pi}R + f_a^2 H^2]$$

we substitute the Einstein field equation (its trace) satisfied by the wormhole instanton solution,

$$R = -\frac{16\pi}{M_p^2} f_a^2 H^2 \tag{22}$$

to obtain $\Big($using $\quad H = h(\tau)\epsilon = \frac{n}{f_a^2 a^3(\tau)}\epsilon \quad$ and $\quad \int_M d^4x\sqrt{g} = \int_0^\infty d\tau \int_{S^3}\epsilon = 2\pi^2 \int_0^\infty d\tau a^3(\tau)\Big)$

$$\begin{aligned} I_{EAT}(\text{instanton}) &= 2f_a^2 \int_M d^4x\sqrt{g}H^2 = \frac{24\pi^2 n^2}{f_a^2}\int_0^\infty d\tau \frac{1}{a^3(\tau)} \\ &= \frac{6\pi^3 n^2}{f_a^2 r^2} = \frac{3\pi^2}{8} r^2 M_p^2 = \frac{3\pi^2}{8}(\frac{r}{l_p})^2 = \frac{3\pi^{5/2} n M_p}{2f_a}. \end{aligned} \tag{23}$$

Recall that the amplitude for the tunnelling from R^3 to $R^3 \oplus S^3$, namely the "baby universe nucleation rate" is proportional to

$$e^{-I_{EAT}(\text{instanton})} = \exp\left[-\frac{3\pi^2}{8}(\frac{r}{l_p})^2\right] \tag{24}$$

where $l_p = M_p^{-1}$ is the Planck length. Thus, first of all, we see that fortunately the rate for the nucleation of the baby universes (or wormholes) with size larger than the Planck length ($r > l_p$) is highly suppressed. Also, we can see that a typical baby universe (or wormhole) will have a radius of order $r{\sim}\sqrt{\frac{8}{3\pi^2}}l_p$. Another quantity which is important in determining the effects of wormhole instantons on the low energy physics is the axion charge carried away by the baby universe. This has the typical value of $n \sim (\frac{2f_a}{3\pi^{5/2}M_p})$. Finally, when the wormhole instantons or equivalently nucleated baby universes are widely separated, the "dilute instanton gas approximation", in which the interactions between instantons can be ignored, can be valid to be used. For the sake of completeness, next we also consider the physics of classical wormhole solution resulting from an alternative gauge choice for the lapse function $N(\tau)$.

(2) With the conformal-time gauge fixing $N(\tau) = a(\tau)$:

$$(\frac{a'}{a})^2 - 1 = -\frac{r^4}{a^4}. \qquad \left(r^2 \equiv \frac{4\sqrt{\pi}n}{M_p f_a} = a^2(\tau = 0)\right) \tag{25}$$

$$\tag{26}$$

which yields, upon integration,

$$\begin{aligned} a(\tau) &= r[cosh(2\tau)]^{1/2}, \\ h(\tau) &= \frac{n}{f_a^2 a^3(\tau)} = \frac{n}{f_a^2 r^3}[cosh(2\tau)]^{-3/2}. \end{aligned} \tag{27}$$

Note the asymptotic behavior of this Euclidean wormhole solution

$$\begin{aligned} a(\tau) &\to \frac{r}{\sqrt{2}}e^{\tau} \qquad (\text{as} \quad \tau \to \infty), \\ &\to \frac{r}{\sqrt{2}}e^{-\tau} \qquad (\text{as} \quad \tau \to -\infty). \end{aligned}$$

Thus again, this wormhole solution represents a configuration in which two asymptotically-Euclidean regions are connected by a wormhole with throat (or neck) whose cross sections are S^3's with minimum radius $a(\tau = 0) = r = (\frac{4\sqrt{\pi}n}{M_p f_a})^{1/2}$. Next, as before, we evaluate the

(wormhole) instanton action I_{EAT}(instanton), namely the minimum Euclidean action of the instanton configuration which makes dominant contribution to the tunnelling amplitude, i.e., baby universe nucleation rate. And this amounts to substituting the Einstein field equation (its trace) satisfied by the wormhole instanton solution into the Euclidean Einstein-antisymmetric tensor theory action as we did before. Now for the case at hand where we take the "conformal time gauge" for the lapse function, $N(\tau) = a(\tau)$, using $H = h(\tau)\epsilon = \frac{n}{f_a^2 a^3(\tau)}\epsilon$ and $\int_M d^4x\sqrt{g} = \int_0^\infty d\tau N(\tau) \int_{S^3} \epsilon = 2\pi^2 \int_0^\infty d\tau N(\tau) a^3(\tau)$, we obtain

$$\begin{aligned} I_{EAT}(\text{instanton}) &= 2f_a^2 \int_M d^4x\sqrt{g} H^2 = \frac{24\pi^2 n^2}{f_a^2} \int_0^\infty d\tau N(\tau) a^3(\tau) \frac{1}{a^6(\tau)} \\ &= \frac{24\pi^2 n^2}{f_a^2} \int_0^\infty \frac{d\tau}{a^2(\tau)} = \frac{6\pi^3 n^2}{f_a^2 r^2} = \frac{3\pi^2}{8}(\frac{r}{l_p})^2 = \frac{3\pi^{5/2} n M_p}{2f_a}. \end{aligned} \tag{28}$$

Notice here that although the form of spacetime metric $a(\tau)$ and the Kalb-Ramond field strength $h(\tau)$ solution are different for two different gauge choices $N(\tau) = 1$ and $N(\tau) = a(\tau)$, the Euclidean instanton action evaluated at these wormhole instanton configurations remains the same indicating the same tunnelling process. Namely, since the quantity $\sim \exp[-I_{EAT}(\text{instanton})]$ represents the semi-classical approximation to the baby universe nucleation rate, it is a physical observable which should have manifest gauge-invariance. The relevant gauge freedom for the case at hand is the arbitrariness in choosing the lapse $N(\tau)$ and we have just confirmed this gauge-invariance. Indeed, this gauge-invariance of the instanton action I_{EAT}(instanton) and hence that of the semi-classical baby universe nucleation rate can be generally displayed as follows; from the general form of the time-time component of Einstein equations

$$(\frac{a'}{a})^2 - \frac{N^2}{a^2} = -r^4 \frac{N^2}{a^6}, \tag{29}$$

we get

$$d\tau = \frac{a^2 da}{N\sqrt{a^4 - r^4}}. \tag{30}$$

Meanwhile, the general form of the Euclidean instanton action is given by (using $\int_M d^4x\sqrt{g} = 2\pi^2 \int_0^\infty d\tau N a^3$)

$$I_{EAT}(\text{instanton}) = 2f_a^2 \int_M d^4x\sqrt{g}H^2 \qquad (31)$$
$$= \frac{24\pi^2 n^2}{f_a^2}\int_0^\infty \frac{N}{a^3}d\tau.$$

Thus by using eq.(30), we finally obtain

$$I_{EAT}(\text{instanton}) = \frac{24\pi^2 n^2}{f_a^2}\int_r^\infty \frac{da}{a\sqrt{a^4 - r^4}} = \frac{6\pi^3 n^2}{f_a^2 r^2}. \qquad (32)$$

We now end this section with some comments. As was noted by Rey [4], the axionic instantons (or wormholes) are characterized by their axion charges n which are necessary for their stability. Therefore, if we call the axionic instanton with the axionic charge $+n$ as an instanton, then we may identify that with the axionic charge $-n$ as its "anti-instanton". Then denoting an instanton with axion charge $+n$ and an anti-instanton with $-n$ by I_n and I_{-n} respectively, we may speculate the pair-annihilation-type process such as

$$I_n + I_{-n} \quad \leftrightarrow \quad \text{(flat spacetime)}.$$

Indeed, the possibility of this process is supported by the fact that for cases when the dilute gas approximation is valid, $I_{EAT}(\text{instanton}) + I_{EAT}(\text{anti-instanton}) = Cn + C(-n) = 0$ (where $C \equiv \frac{3\pi^{5/2}M_p}{2f_a}$), namely the total action of the instanton-anti-instanton system is zero which is the action of the vacuum, i.e., flat spacetime. And this statement may remain true for any other theory involving wormhole solution which is stabilized by global charges of the underlying physics. Next, if we carefully distinguish between Fig.1 and 2 such that Fig.1 depicts "wormhole" configuration and Fig.2, "instanton" configuration, the wormhole configuration can be identified with a "bounce" solution of $R^3 \to R^3 \oplus S^3 \to R^3$. Then a wormhole with axionic charge $+n$ is the double of two axionic instantons with charge $+n$ and $-n$ respectively or equivalently of two oppositely-oriented instantons sewed together along the (Euclidean) spacelike boundaries S^3's. Finally, taking our $SO(4)$-symmetric homogeneous and isotropic wormhole and axion field solution as the ground state (i.e., maximally-symmetric) solution, one may wish to look for exited wormhole and axion field solutions with, say, slightly broken $SO(4)$-symmetry. For example, one may try with

the wormhole solution ansätz being given by the Bianchi type-IX metric [13] which is still homogeneous but not exactly isotropic.

III. The fermion zero modes and their effects on low-energy physics

As have been pointed out first by Hosoya and Ogura [5], but in a different context where they studied wormhole instantons in Einstein-Yang-Mills theory, if there exist fermionic zero modes in a given background of wormhole spacetime, they may have profound effects on low-energy physics presumably in a similar manner the instanton configurations in non-abelian gauge theories do. To name one, one may expect that chirality-changing fermion propagation in the background of a wormhole instanton could arise. In order to see if this kind of intriguing possibility can actually happen in our case of axionic wormhole instanton, we first attempt to investigate the existence of normalizable fermion zero modes in the background of axionic wormhole instantons. Thus far, we have considered the theory of free KR (classical) antisymmeric tensor field $B_{\mu\nu}$ coupled only to gravity. In order to explore the dynamics of fermion field in the background of classical wormhole instanton solutions in Einstein-antisymmetric tensor theory, we need to know the fermion-KR antisymmetric tensor field interaction as well as the fermion-gravity minimal coupling. To our knowledge, however, the fermion-KR antisymmetric tensor gauge field interaction (*not* the Yukawa-type axion-axial fermion current interactions $A(x)\bar{\Psi}\gamma_5\Psi$ or $\partial_\mu A(x)\bar{\Psi}\gamma^\mu\gamma_5\Psi$ in effective Lagrangians of extended standard model based on the gauge group $SU(3)_C\times SU(2)_L\times U(1)_Y\times U(1)_{PQ}$ with $U(1)_{PQ}$ being the anomalous Peccei-Quinn symmetry group) is not known nor has been seriously considered yet. Since the pure KR antisymmetric tensor gauge theory possesses a gauge invariance based on abelian $\Big(U(1)\Big)$ gauge group as mentioned earlier, one can construct a fermion-KR antisymmetric tensor gauge field interaction Lagrangian which has a manifest local $U(1)$ gauge-invariance. Here in the present work, we propose, as one such attempt, a theory of massless fermion-KR antisymmetric tensor field system involving both the local tensor $U(1)_V$ and axial tensor $U(1)_A$ gauge-invariant couplings described by the action in flat Minkowski spacetime

$$S_{KR-F} = \int d^4x[-f_a^2 tr(H_{\mu\nu\lambda}H^{\mu\nu\lambda}) + \bar{\Psi}i\gamma^\mu D_\mu \Psi] \tag{33}$$

with the gauge-covriant derivative being given by

$$\begin{aligned}\gamma^\mu D_\mu &\equiv (\gamma^\mu \partial_\mu - \sigma^{\mu\nu} B^V_{\mu\nu} + \sigma^{\mu\nu}\gamma_5 B^A_{\mu\nu}) \\ &= (\gamma^\mu \partial_\mu - i\gamma^\mu\gamma^\nu B^V_{\mu\nu} + i\gamma^\mu\gamma^\nu\gamma_5 B^A_{\mu\nu}) \\ &= \gamma^\mu(\partial_\mu - i\gamma^\nu B^V_{\mu\nu} + i\gamma^\nu\gamma_5 B^A_{\mu\nu})\end{aligned} \tag{34}$$

where $\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu]$ and we let $B^{VA}_{\mu\nu} \to B^{VA}_{\mu\nu} I$ (with I being the 4×4 identity matrix which explains "tr" in the KR field term in the action above. It is straightforward to check that this action is invariant under the local tensor $U(1)_V$ and axial tensor $U(1)_A$ transformations given by

$$B^{VA}_{\mu\nu} \to B^{VA}_{\mu\nu} + (\partial_\mu \Lambda^{VA}_\nu - \partial_\nu \Lambda^{VA}_\mu) \tag{35}$$

where

$$\Lambda^\mu_{VA}(x) = \frac{1}{2(n-1)}\gamma^\mu \theta_{VA}(x) \qquad (\text{say, in } n-\text{dim.}) \tag{36}$$

along with

$$\Psi \to e^{i\theta_V(x)}\Psi \qquad , \qquad \bar{\Psi} \to \bar{\Psi}e^{-i\theta_V(x)} \tag{37}$$

for local $U(1)_V$ transformation and

$$\Psi \to e^{i\gamma_5\theta_A(x)}\Psi \qquad , \qquad \bar{\Psi} \to \bar{\Psi}e^{i\gamma_5\theta_A(x)} \tag{38}$$

for local $U(1)_A$ transformation. The guideline for our choice of the fermion-KR antisymmetric tensor gauge field interaction given above is as follows; certainly we need minimal coupling in which KR field itself $B_{\mu\nu}(x)$, not its field strength $H_{\mu\nu\lambda}(x)$, is supposed to be present as is obvious from our experience with ordinary vector gauge field theories. Next, since the KR tensor gauge field $B^V_{\mu\nu}(x)$ is antisymmetric under interchange of its indices, antisymmetric fermion tensor current of rank-2, $J^{\mu\nu} = \bar{\Psi}\sigma^{\mu\nu}\Psi$, which appears to be the only

choice available, should couple to it, i.e., $J^{\mu\nu}B^V_{\mu\nu} = \bar{\Psi}\sigma^{\mu\nu}\Psi B^V_{\mu\nu}$. And a similar argument applies to our choice of KR axial tensor gauge field $B^A_{\mu\nu}$-fermion axial tensor current coupling term, $J^{\mu\nu}_5 B^A_{\mu\nu} = \bar{\Psi}\sigma^{\mu\nu}\gamma_5\Psi B^A_{\mu\nu}$. Now, since the examination of the dynamical fermion fields in the background of classical KR antisymmetric tensor field and curved spacetime (i.e., wormhole geometry) is of our present interest in this work, next we consider the case when the gravity is turned on (but just as a "background" field) with the KR antisymmetric tensor field freezing again as a non-dynamical degree. Then the theory of a dynamical fermion field in the background of KR and gravitational field would naturally be described by the action

$$\begin{aligned} S_F &= \int d^4x\ e\ \frac{i}{2}[\bar{\Psi}\gamma^\mu \overrightarrow{\nabla}_\mu \Psi - \bar{\Psi}\gamma^\mu \overleftarrow{\nabla}_\mu \Psi] \\ &= \int d^4x\ e\ \frac{i}{2}[\bar{\Psi}\gamma^A e^\mu_A \overrightarrow{\nabla}_\mu \Psi - \bar{\Psi}\gamma^A e^\mu_A \overleftarrow{\nabla}_\mu \Psi] \end{aligned} \tag{39}$$

where the covariant derivative now generalizes to

$$\begin{aligned} \gamma^\mu \nabla_\mu &\equiv [\gamma^\mu(\partial_\mu - \frac{i}{4}\omega^{AB}_\mu \sigma_{AB}) - \sigma^{\mu\nu}B_{\mu\nu}] \\ &= \gamma^C e^\mu_C[\partial_\mu - \frac{i}{4}\omega^{AB}_\mu \sigma_{AB} - i\gamma^B e^\nu_B B_{\mu\nu}]. \end{aligned} \tag{40}$$

(Here we consider only the KR tensor field - fermion tensor current coupling, $J^{\mu\nu}B^V_{\mu\nu}$, which is of usual relevance.) Then the corresponding Dirac equations for massless fermion field are given by

$$\begin{aligned} &\gamma^C e^\mu_C[\overrightarrow{\partial}_\mu - \frac{i}{4}\omega^{AB}_\mu \sigma_{AB} - i\gamma^B e^\nu_B B_{\mu\nu}]\Psi = 0, \\ &\bar{\Psi}\gamma^C e^\mu_C[\overleftarrow{\partial}_\mu - \frac{i}{4}\omega^{AB}_\mu \sigma_{AB} - i\gamma^B e^\nu_B B_{\mu\nu}] = 0. \end{aligned} \tag{41}$$

In the action and Dirac equations above, $e^A_\mu(x)\Big(e^\mu_A(x)\Big)$ is the "vierbein" (and it's inverse) defined by $g_{\mu\nu}(x) = \delta_{AB}e^A_\mu(x)e^B_\nu(x)$ and $e^A_\mu e^\mu_B = \delta^A{}_B$, $e^\mu_A e^A_\nu = \delta^\mu{}_\nu$ and $e \equiv (det\ e^A_\mu)$. Thus the Greek indices μ, ν refer to coordinate basis while the Roman indices A,B = 0,1,2,3 refer to non-coordinate basis. Now $\gamma^\mu(x) = e^\mu_A(x)\gamma^A$ is the curved spacetime γ-matrices obeying $\{\gamma^\mu(x), \gamma^\nu(x)\} = -2g_{\mu\nu}(x)$ with γ^A being the usual flat spacetime γ-matrices. Next $(\partial_\mu - \frac{i}{4}\omega^{AB}_\mu \sigma_{AB})$ is then the Lorentz covariant derivative with ω^{AB}_μ being the spin connection

and σ_{AB} being the $SO(3,1)$ group generator in the spinor representation given respectively by

$$\begin{aligned}\omega^A_{\mu B} &= -e^\nu_B(\partial_\mu e^A_\nu - \Gamma^\lambda_{\mu\nu}e^A_\lambda),\\ \sigma^{AB} &= \frac{i}{2}[\gamma^A,\gamma^B].\end{aligned} \tag{42}$$

With this general preparation, now we turn to the examination of the existence of fermion zero modes in the background of axionic wormhole solutions in Einstein-antisymmetric tensor theory. As before, we treat the problem in two different choices of gauge associated with the time reparametrization invariance, $N(\tau)=1$ and $N(\tau)=a(\tau)$ one by one.

(1) With the gauge choice $N(\tau)=1$:

As discussed earlier, in this gauge, the axionic wormhole spacetime is described by the Euclidean (spatially-closed) FRW metric given by

$$\begin{aligned}ds^2 &= d\tau^2 + a^2(\tau)\sigma^a\otimes\sigma^a \\ &= g_{\mu\nu}dx^\mu dx^\nu = \delta_{AB}e^A\otimes e^B\end{aligned} \tag{43}$$

with the scale factor $a(\tau)$ being given by

$$\tau = \frac{r}{\sqrt{2}}F[cos^{-1}(\frac{r}{a}),\frac{1}{\sqrt{2}}] - \sqrt{2}rE[cos^{-1}(\frac{r}{a}),\frac{1}{\sqrt{2}}] + \frac{1}{a}\sqrt{a^4-r^4} \qquad \Big(a(0)=r\Big). \tag{44}$$

The non-coordinate basis 1-forms are read off as

$$e^A = \{e^0 = d\tau,\ e^a = a(\tau)\sigma^a\} \tag{45}$$

where $\{\sigma^a\}$ $(a=1,2,3)$ form a basis on the three-sphere S^3 satisfying the $SU(2)$ "Maurer-Cartan" structure equation

$$d\sigma^a = \frac{1}{2}\epsilon^{abc}\sigma^b\wedge\sigma^c \tag{46}$$

and can be represented in terms of 3-Euler angles $0\le\theta\le\pi$, $0\le\phi\le 2\pi$ and $0\le\psi\le 4\pi$, parametrizing S^3

$$\sigma^1 = cos\psi d\theta + sin\psi sin\theta d\phi,$$
$$\sigma^2 = sin\psi d\theta - cos\psi sin\theta d\phi, \tag{47}$$
$$\sigma^3 = d\psi + cos\theta d\phi.$$

Then the associated vierbein and its inverse are found to be (using $e^A = e^A_\mu dx^\mu$, $x^\mu = (\tau, \theta, \phi, \psi)$)

$$e^A_\mu = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & acos\psi & asin\psi sin\theta & 0 \\ 0 & asin\psi & -acos\psi sin\theta & 0 \\ 0 & 0 & acos\theta & a \end{pmatrix} \quad , \quad e^\mu_A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{a}cos\psi & \frac{1}{a}sin\psi & 0 \\ 0 & \frac{sin\psi}{asin\theta} & \frac{-cos\psi}{asin\theta} & 0 \\ 0 & \frac{-sin\psi cos\theta}{asin\theta} & \frac{cos\psi cos\theta}{asin\theta} & \frac{1}{a} \end{pmatrix}. \tag{48}$$

Next, we obtain the spin-connection 1-forms, using the Cartan's 1st structue equation (i.e., torsion-free condition)

$$de^A + \omega^A_{\ B} \wedge e^B = 0 \tag{49}$$

with the help of Maurer-Cartan structure equation given earlier, to be (in Euclidean signature)

$$\omega^a_{\mu 0} = -\omega^0_{\mu a} = (\frac{a'}{a})e^a_\mu \quad , \quad \omega^a_{\mu b} = -\omega^b_{\mu a} = \frac{1}{2a}\epsilon^{abc}e^c_\mu. \tag{50}$$

The KR antisymmetric tensor field $B_{\mu\nu}$ can be given in this non-coordinate basis, B_{AB}, as well. Recall that the KR antisymmetric tensor field strength giving the axionic wormhole instanton solution was given in coordinate basis by

$$H_{\mu\nu\lambda} = h(\tau)\epsilon_{\mu\nu\lambda} = \frac{n}{f_a^2 a^3(\tau)}\epsilon_{\mu\nu\lambda} \tag{51}$$

where $\mu, \nu, \lambda \neq \tau$. In order to find the associated KR tensor field itself, we choose the gauge for which $B_{\mu\nu} = B_{\mu\nu}(\tau)$, i.e., $B_{\mu\nu}$ is a function of Euclidean time alone and use its relation to its field strength $H = dB$ where

$$H = \frac{1}{3!}h(\tau)\epsilon_{ABC}e^A \wedge e^B \wedge e^C = \frac{1}{3!}h(\tau)\epsilon_{abc}e^a \wedge e^b \wedge e^c,$$
$$B = \frac{1}{2!}B_{AB}(\tau)e^A \wedge e^B = \frac{1}{2!}B_{ab}(\tau)e^a \wedge e^b. \tag{52}$$

Note here that we used $e^A = e^A_\mu dx^\mu$, $\epsilon_{ABC} = e^\mu_A e^\nu_B e^\lambda_C \epsilon_{\mu\nu\lambda}$ and since $\mu, \nu, \lambda \neq \tau$ and the FRW-metric is devoid of time-space off-diagonal components, $A, B, C \to a, b, c \neq 0$. Then in non-coordinate basis, the KR antisymmetric tensor field is found to be

$$B_{ab}(\tau)\epsilon^{acd} = \frac{1}{3}h(\tau)a(\tau)\epsilon^{bcd} = \frac{n}{3f_a^2}\frac{1}{a^2(\tau)}\epsilon^{bcd}. \tag{53}$$

Now, consider the Dirac equation for massless fermion fields in the background of KR anti-symmetric tensor field and curved spacetime obtained earlier

$$\gamma^C e^\mu_C[\partial_\mu - \frac{i}{4}\omega^{AB}_\mu \sigma_{AB} - i\gamma^B e^\nu_B B_{\mu\nu}]\Psi = 0.$$

For the case at hand,

$$\begin{aligned} \gamma^C e^\mu_C \partial_\mu &= \gamma^0 \partial_\tau, \\ \gamma^C e^\mu_C \omega^{AB}_\mu \sigma_{AB} &= 6i\gamma^0(\frac{a'}{a}) + i(\frac{1}{2a})\epsilon_{abc}\gamma^a\gamma^b\gamma^c, \\ \gamma^\mu\gamma^\nu B_{\mu\nu} &= \gamma^a\gamma^b B_{ab} \end{aligned} \tag{54}$$

where we used $\omega^{a0}_\mu e^\mu_b = (\frac{a'}{a})\delta^a_b$, $\omega^{ab}_\mu e^\mu_c = (\frac{1}{2a})\epsilon^{abc}$. Further, assuming that the fermion field depends only on the Euclidean time τ and setting

$$\Psi(\tau) = a^{-\frac{3}{2}}(\tau)\tilde{\Psi}(\tau), \tag{55}$$

the Dirac equation above reduces to

$$[\partial_\tau + \frac{3}{4}\gamma_5\frac{1}{a(\tau)} - i\gamma^0\gamma^a\gamma^b B_{ab}(\tau)]\tilde{\Psi}(\tau) = 0 \tag{56}$$

where $\gamma_5 = \gamma^0\gamma^1\gamma^2\gamma^3$ is the Euclidean γ_5-matrix. Here, noticing $\gamma^0\gamma^a\gamma^b B_{ab} = \gamma_5\epsilon^{abc}\gamma^a B_{bc}$ and thus if we set, using eq.(53),

$$\epsilon^{abc}\gamma^a B_{bc}(\tau) = M\frac{1}{a^2(\tau)} \tag{57}$$

(thus here M is a (4×4) matrix whose precise form is not of direct relevance for the discussion below), its solution is given by

$$\tilde{\Psi}(\tau) = exp[\pm\{\frac{3}{4}\int_0^\tau \frac{d\tau'}{a(\tau')} - iM\int_0^\tau \frac{d\tau'}{a^2(\tau')}\}]\ u \tag{58}$$

where u denotes the constant basis spinor. Thus the solution to the massless Dirac equation is found to be

$$\Psi(\tau) = \frac{1}{a^{\frac{3}{2}}(\tau)}\tilde{\Psi}(\tau).$$

Here the $\pm$ signs refer to each of the two chiralities of the fermion field. Note that the τ-integration in the exponent is finite due to finite integration range and M involves complex matrix. Thus owing to the convergence factor $a^{-3/2}(\tau)$ which dies out as $\tau \to \pm\infty$ as we observed earlier, this solution to the massless Dirac equation, namely the fermionic zero mode is most probably normalizable. This means the existence of two normalizable fermion zero modes.

(2) With the gauge choice $N(\tau) = a(\tau)$:

In this "conformal-time gauge", the Euclidean FRW-metric for the $SO(4)$-symmetric axionic wormhole spacetimes takes the form given by

$$\begin{aligned} ds^2 &= a^2(\tau)[d\tau^2 + \sigma^a \otimes \sigma^a] \\ &= g_{\mu\nu}dx^\mu dx^\nu = \delta_{AB}e^A \otimes e^B \end{aligned} \tag{59}$$

with the scale factor $a(\tau)$ being given by

$$a(\tau) = r[cosh(2\tau)]^{1/2}. \tag{60}$$

The non-coordinate basis 1-forms are immediately read off as

$$e^A = \{e^0 = a(\tau)d\tau, e^a = a(\tau)\sigma^a\} \tag{61}$$

with $\{\sigma^a\}$ $(a = 1, 2, 3)$ again being the left-invariant 1-forms on S^3 satisfying Maurer-Cartan structure equation given earlier. Then the associated vierbein and inverse vierbein are found as

$$e^A_\mu = a(\tau)\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & cos\psi & sin\psi sin\theta & 0 \\ 0 & sin\psi & -cos\psi sin\theta & 0 \\ 0 & 0 & cos\theta & 1 \end{pmatrix} \quad , \quad e^\mu_A = \frac{1}{a(\tau)}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & cos\psi & sin\psi & 0 \\ 0 & \frac{sin\psi}{sin\theta} & \frac{-cos\psi}{sin\theta} & 0 \\ 0 & \frac{-sin\psi cos\theta}{sin\theta} & \frac{cos\psi cos\theta}{sin\theta} & 1 \end{pmatrix}. \tag{62}$$

Next, we obtain the spin-connection 1-form using the Cartan's 1st structure equation and the $SU(2)$ Maurer-Cartan structure equation given earlier and they are

$$\omega^a_{\mu 0} = -\omega^0_{\mu a} = (\frac{a'}{a^2})e^a_\mu \quad , \quad \omega^a_{\mu b} = -\omega^b_{\mu a} = \frac{1}{2a}\epsilon^{abc}e^c_\mu. \tag{63}$$

And as we did before in the case when we chose the gauge $N(\tau) = 1$, we can obtain the KR antisymmetric tensor field in non-coordinate basis to be

$$B_{ab}(\tau)\epsilon^{acd} = \frac{n}{3f_a^2}\frac{1}{a^2(\tau)}\epsilon^{bcd} \tag{64}$$

which turns out to be the same as that in the case with the gauge choice $N(\tau) = 1$. Then, again consider the Dirac equation for massless fermion fields in the background of axionic wormhole spacetime comprised of the KR antisymmetric tensor field and the metric field solution given earlier. For the present case in which we choose the gauge $N(\tau) = a(\tau)$,

$$\gamma^C e^\mu_C \partial_\mu = \frac{1}{a}\gamma^0\partial_\tau, \tag{65}$$
$$\gamma^C e^\mu_C \omega^{AB}_\mu \sigma_{AB} = 6i\gamma^0(\frac{a'}{a^2}) + i(\frac{1}{2a})\epsilon_{abc}\gamma^a\gamma^b\gamma^c,$$

where we used $\omega^{a0}_\mu e^\mu_b = (\frac{a'}{a^2})\delta^a_b$, $\omega^{ab}_\mu e^\mu_c = (\frac{1}{2a})\epsilon^{abc}$. Again, assuming that the fermion field has dependence only on the Euclidean time τ and setting

$$\Psi(\tau) = a^{-\frac{3}{2}}(\tau)\tilde{\Psi}(\tau),$$

the Dirac equation becomes

$$[\partial_\tau + \frac{3}{4}\gamma_5 - i\gamma^0\gamma^a\gamma^b B_{ab}(\tau)a(\tau)]\tilde{\Psi}(\tau) = 0$$

Now noticing again $\gamma^0\gamma^a\gamma^b B_{ab} = \gamma_5\epsilon^{abc}\gamma^a B_{bc}$ and using eq.(64), if we set $\epsilon^{abc}\gamma^a B_{bc}(\tau) = M\frac{1}{a^2(\tau)}$ as before, its solution is given by

$$\tilde{\Psi}(\tau) = \exp[\pm\{\frac{3}{4}\tau - iM\int_0^\tau \frac{d\tau'}{a(\tau')}\}]u \tag{66}$$

with the $\pm$ signs referring to each chirality of the fermion field and hence the solution to the massless Dirac equation is found to be

$$\Psi(\tau) = \frac{1}{a^{\frac{3}{2}}(\tau)}\tilde{\Psi}(\tau).$$

Now we can invoke the same argument as the one we employed before to establish the normalization of this solution to the massless Dirac equation. Namely, since the τ-integration range in the exponent is finite and M involves complex matrix, this fermion zero mode is most probably normalizable particularly owing to the obvious convergence factor $a^{-3/2}(\tau)$ which dies out as $\tau \to \pm\infty$. And again, this means the existence of two normalizable fermion zero modes. As commented earlier in this section, the consequence of the existence of normalizable fermion zero modes in the background of axionic wormhole spacetime may be significant. Firstly, as has been pointed out by Rey [5] in the context of wormhole solutions in Einstein-Yang-Mills theory, the existence of fermion zero modes would affect wormhole interactions. Namely, one may expect that, the fermion zero modes, upon integration, would yield a long-range confining interaction between the axionic wormholes. Secondly, we have observed in the analysis that the solutions to the massless Dirac equation, regardless of the gauge choice $N(\tau) = 1$ or $a(\tau)$, are symmetric with respect to the chirality flip. And this may signal that the axionic wormhole instantons would not induce the chirality-changing fermion propagation in a manner similar to instantons in non-abelian gauge theories typically do. As is well-known, in non-abelian gauge theories, chirality-changing fermion propagation is attributed to non-trivial instanton configuration or non-vanishing instanton number which, in turn, is directly related to the chiral anomaly. Therefore, the absence of chirality-changing fermion propagation seems to imply that the local, axial KR tensor gauge symmetry introduced earlier is not anomalous.

IV. Quantum Wormholes in Einstein-antisymmetric Tensor Theory

We would like to construct and study a minisuperspace quantum cosmology model based on Einstein-KR antisymmetric tensor theory (or axionic gravity theory) generally in the presence of the cosmological constant Λ described by the action

$$S_{EAT} = \int_M d^4x\sqrt{g}[\frac{M_p^2}{16\pi}R - \Lambda - f_a^2 H_{\mu\nu\lambda}H^{\mu\nu\lambda}] + \int_{\partial M} d^3x\sqrt{h}\frac{M_p^2}{8\pi}(K - K_0), \tag{67}$$

$$I_{EAT} = \int_M d^4x\sqrt{g}[\Lambda - \frac{M_p^2}{16\pi}R + f_a^2 H_{\mu\nu\lambda}H^{\mu\nu\lambda}] - \int_{\partial M} d^3x\sqrt{h}\frac{M_p^2}{8\pi}(K - K_0) \quad (68)$$

in Lorentzian and Euclidean signatures respectively. As for our approach, we choose to take the avenue of canonical quantum cosmology based on Arnowitt-Deser-Misner (ADM)'s (3+1) space-plus-time split formulation [13-15]. As usual, then, in order to render the system tractable, we reduce the infinite-dimensional superspace down to a 2-dimensional minisuperspace by assuming that the 4-dimensional spacetime has the geometry of spatially-closed ($k = +1$) FRW-metric. The geometry of its spatial section is, then, that of S^3 and hence it has $SO(4)$-symmetry. Then, since the spatial geometry is taken to possess the $SO(4)$-symmetry, the matter field, i.e., the antisymmetric tensor field (Kalb-Ramond field) defined on it should have the same $SO(4)$-symmetry. Thus we can choose the following $SO(4)$-symmetric ansätz for the metric and antisymmetric tensor field

$$\begin{aligned} ds^2 &= \sigma^2[-N^2(t)dt^2 + a^2(t)d\Omega_3^2] \\ &= \sigma^2[-N^2(t)dt^2 + a^2(t)\sigma^a \otimes \sigma^a] = \eta_{AB}e^A e^B \\ \text{where} \quad & e^A = \{e^0 = \sigma N(t)dt,\ e^a = \sigma a(t)\sigma^a\} \quad (a = 1,2,3) \end{aligned} \quad (69)$$

and

$$\begin{aligned} H &= \frac{h(t)}{f_a}\epsilon \\ \text{where} \quad \epsilon &= \frac{1}{3!}\epsilon_{\mu\nu\lambda}dx^\mu \wedge dx^\nu \wedge dx^\lambda = \sqrt{h}d^3x, \\ \int_{S^3}\epsilon &= \int_{S^3} d^3x\sqrt{h} = 2\pi^2 a^3(t)\sigma^4 \end{aligned} \quad (70)$$

where $\sigma^2 \equiv (\frac{2}{3\pi M_p^2})$ has now been introduced for convenience and $d\Omega_3^2$ denotes the line element on S^3. Note here that in this quantum treatment, we choose the $SO(4)$-symmetric ansätz for the KR antisymmetric tensor field strength slightly differently from that in the previous classical treatment in which we took $H = h(t)\epsilon$. Again, the left-invariant 1-forms σ^a (a = 1,2,3) form a basis 1-form on S^3 satisfying the $SU(2)$ Maurer-Cartan structure equation in eq.(46) and can be represented in terms of 3-Euler angles ($\theta,\ \phi,\ \psi$) parametrizing S^3 as given in eq.(47).

In order to eventually write the Lorenzian action in terms of these $SO(4)$-symmetric ansätz for the metric and antisymmetric tensor field, consider

$$\int d^4x\sqrt{g} = \int dtN(\int_{S^3} d^3x\sqrt{h}) = 2\pi^2\sigma^4\int dtNa^3$$

and for the choice of the ansätz $H_{\mu\nu\lambda} = \frac{h(t)}{f_a}\epsilon_{\mu\nu\lambda}$

$$H_{\mu\nu\lambda}H^{\mu\nu\lambda} = \sigma\frac{h^2(t)}{f_a^2}. \tag{71}$$

Here, first notice that the action term of the Kalb-Ramond antisymmetric tensor field $H_{\mu\nu\lambda}H^{\mu\nu\lambda}$ involves no kinetic term but only the potential term. Next, recall that since $H_{\mu\nu\lambda}$ is the field strength for the Kalb-Ramond antisymmetric tensor $B_{\mu\nu}$, it can be written as $H = dB$. It, then, immediately follows that $H_{\mu\nu\lambda}$ must satisfy the Bianchi identity $dH = 0$ which, in our choice of the $SO(4)$-symmetric FRW-metric, amounts to taking $h(t) = n/a^3(t)$ with n being a constant to be fixed later. Namely, recall that the SO(4)-symmetric ansätz for the KR antisymmetric tensor field strength $H_{\mu\nu\lambda}$ (or $h(t)$) given above remains valid off-shell as well as on-shell as we stressed earlier in the classical treatment of the system. And it is because we obtained it simply from the definition and the Bianchi identity which are certainly bottomline conditions that should be met in quantum formulations as well.

Now, we are ready to write the (Lorentzian) action for Einstein-KR antisymmetric tensor theory in terms of the $SO(4)$-symmetric ansätz for the metric and matter field.

$$\begin{aligned} S_G &= \int d^4x\sqrt{g}[\frac{M_p^2}{16\pi}R - \Lambda] = \frac{1}{2}\int dtNa^3[-\lambda + \{\frac{1}{a^2} - (\frac{\dot{a}}{na})^2\}], \\ S_{AT} &= \int d^4x\sqrt{g}[-f_a^2 H_{\mu\nu\lambda}H^{\mu\nu\lambda}] \\ &= (2\pi^2\sigma^4)\int dtNa^3(-6h^2(t)) = \frac{1}{2}\int dtNa^3[-H^2(t)]. \end{aligned} \tag{72}$$

Thus

$$\begin{aligned} S_{EAT} &= S_G + S_{AT} \\ &= \frac{1}{2}\int dtNa^3[-\lambda + \{\frac{1}{a^2} - (\frac{\dot{a}}{na})^2\} - H^2(t)] \end{aligned} \tag{73}$$

where we introduced $\lambda \equiv 16\Lambda/9M_p^4$ and $H(t) \equiv \sqrt{24}\pi\sigma^2 h(t) = \sqrt{24}\pi\sigma^2 n/a^3(t)$ and the "overdot" denotes the derivative with respect to the Lorentzian time t.

As mentioned earlier, the definition of $H_{\mu\nu\lambda}$, namely the field strength of the KR antisymmetric tensor, $H = dB$, automatically demands it to satisfy the Bianchi identity $dH = 0$ which amounts to taking $H_{\mu\nu\lambda} = \frac{h(t)}{f_a}\epsilon_{\mu\nu\lambda} = \frac{n}{f_a}\frac{1}{a^3(t)}\epsilon_{\mu\nu\lambda}$ in this $SO(4)$-symmetric system. Here, now we fix the constant n such that $H(t) = \frac{r^2}{a^3(t)}$, i.e., $r^2 = \sqrt{24}\pi\sigma^2 n = 4\sqrt{6}n/3M_p^2$. Then finally the action for the Einstein-KR antisymmetric tensor theory takes the form

$$\begin{aligned} S_{EAT} &= \frac{1}{2}\int dt N a^3[-\lambda + \{\frac{1}{a^2} - (\frac{\dot{a}}{na})^2\} - \frac{r^4}{a^6}] = \int dt L_{ADM}, \\ I_{EAT} &= \frac{1}{2}\int dt N a^3[\lambda - \{\frac{1}{a^2} + (\frac{\dot{a}}{na})^2\} + \frac{r^4}{a^6}] \end{aligned} \tag{74}$$

in Lorentzian and in Euclidean signature respectively.

Namely, the action for Einstein-KR antisymmetric tensor theory becomes effectively that for pure gravity system with an additional potential term $\sim r^4/a^6$. Next, we obtain the Hamiltonian of this Einstein-KR antisymmetric tensor field system via the usual Legendre transformation. To this end, we first identify the momentum conjugate to the scale factor a as

$$p_a = \frac{\partial L_{ADM}}{\partial \dot{a}} = \frac{a}{N}(-\dot{a}). \tag{75}$$

Thus, from

$$\begin{aligned} S_{EAT} &= \int dt L_{ADM} \\ &= \int dt(p_a\dot{a} - H_{ADM}) = \int [p_a da - (NH_0 + N_i H^i)dt] \end{aligned} \tag{76}$$

where $H_{ADM} = NH_0 + N_i H^i$, it follows that

$$\begin{aligned} \frac{\delta S_{EAT}}{\delta N} &= \frac{1}{2}a^3[-\lambda - \frac{r^4}{a^6} + \frac{1}{a^2} + \frac{\dot{a}^2}{N^2a^2}] \\ &= \frac{1}{2a}[p_a^2 - \{\lambda a^4 - a^2 + \frac{r^4}{a^2}\}]. \end{aligned} \tag{77}$$

Namely, the Hamiltonian of the system is found to be

$$\begin{aligned} H_0 = -\frac{\delta S_{EAT}}{\delta N} &= \frac{1}{2a}[-p_a^2 + \{\lambda a^4 - a^2 + \frac{r^4}{a^2}\}] \\ &\equiv \frac{1}{2a}[-p_a^2 + U(a)] \\ \text{where} \quad U(a) &= \lambda a^4 - a^2 + \frac{r^4}{a^2}. \end{aligned} \tag{78}$$

General relativity is one of the most well-known constrained system. The invariance of the system under the 4-dim. diffeomorphisms (consisting of the time-reparametrization and the 3-dim. general coordinate transformations of the spacelike hypersurface) leads to the emergence of 4-constraint equations. Of them, we need not explicitly impose the 3-momentum constraint equations since we already have taken the $N^i = 0$ gauge which amounts to assuming the $SO(4)$-symmetric spatially-closed FRW-metric. Thus, we only need to impose the Hamitonian constraint $H_0 = 0$. The classical Hamiltonian constraint now reads

$$H_0 = \frac{1}{2a}[-p_a^2 + \{\lambda a^4 - a^2 + \frac{r^4}{a^2}\}] = 0. \tag{79}$$

Now, in order to quantize this Einstein-KR antisymmetric tensor system, we need to turn to the "Dirac quantization procedure" for the constrained system. According to the Dirac quantization procedure, the invariance in the action of the theory under the 4-dim. diffeomorphism is secured by demanding that the physical (universe) wave function Ψ be annihilated by "operator versions" of the 4-constraints. Therefore, the classical Hamiltonian constraint above turns into its quantum version, namely the Wheeler-DeWitt equation given by

$$\hat{H}_0(p_a = -i\frac{\partial}{\partial a})\Psi[a] = 0. \tag{80}$$

In order to obtain the correct form of this Wheeler-DeWitt equation, we first examine the structure of the classical Hamiltonian

$$\begin{aligned} H_0 &= \frac{1}{2a}[-p_a^2 + \{\lambda a^4 - a^2 + \frac{r^4}{a^2}\}] = 0. \\ &= T + V \equiv \frac{1}{2}G^{\alpha\beta}\Pi_\alpha\Pi_\beta + V. \end{aligned} \tag{81}$$

Here, one can readily read off the "minisuperspace metric" $G_{\alpha\beta}$ as

$$G_{\alpha\beta} = -a\delta_{\alpha\beta} \quad , \quad G^{\alpha\beta} = -\frac{1}{a}\delta^{\alpha\beta} \tag{82}$$

with $\gamma^\alpha = a$ and $\Pi_\alpha = p_a$ being the minisuperspace variable and its conjugate momentum respectively. Now, by the usual substitution,

$$G^{\alpha\beta}\Pi_\alpha\Pi_\beta \quad \rightarrow \quad -\nabla^2 \tag{83}$$

with $\nabla^2 = \frac{1}{\sqrt{G}}\frac{\partial}{\partial\gamma^\alpha}(\sqrt{G}G^{\alpha\beta}\frac{\partial}{\partial\gamma^\beta}) = -\frac{1}{a}\frac{\partial}{\partial a}(\frac{\partial}{\partial a})$, finally we arrive at the Wheeler-DeWitt equation

$$\hat{H}_0\Psi = \frac{1}{2}\Big[\frac{1}{a}\frac{\partial}{\partial a}(\frac{\partial}{\partial a}) + \frac{1}{a}U(a)\Big]\Psi[a] = 0. \tag{84}$$

Note, here, that the minisuperspace metric $G_{\alpha\beta}(\gamma)$ is generally a function of minisuperspace variables γ^α. Therefore, in passing from classical to quantum version there arises the "ambiguity in operator ordering" problem. Thus, although it is not the most general form, by rewriting

$$\frac{\partial}{\partial a}(\frac{\partial}{\partial a}) \quad \rightarrow \quad \frac{1}{a^p}\frac{\partial}{\partial a}(a^p\frac{\partial}{\partial a}) \tag{85}$$

as suggested by Hartle and Hawking [14], one can partly encompass the "operator-ordering"problem. Finally, the Wheeler-DeWitt equation generally takes the form

$$\frac{1}{2}\Big[\frac{1}{a^p}\frac{\partial}{\partial a}(a^p\frac{\partial}{\partial a}) + U(a)\Big]\Psi[a] = 0 \tag{86}$$

where "p" denotes an index representing the ambiguity in "operator-ordering" and $U(a) = (\lambda a^4 - a^2 + \frac{r^4}{a^2})$. First of all, in order to have some insight into the behavior of the solution of this Wheeler-DeWitt equation we assign the "normal" sign to the "kinetic" energy term to get

$$\frac{1}{2}\Big[\frac{-1}{a^p}\frac{\partial}{\partial a}(a^p\frac{\partial}{\partial a}) + \tilde{U}(a)\Big]\Psi[a] = 0. \tag{87}$$

Then the "potential" energy can be identified with

$$\tilde{U}(a) = -U(a) = (a^2 - \lambda a^4 - \frac{r^4}{a^2}). \tag{88}$$

The Fig.3(4) given displays the plot of "potential"energy as it appears in the WD equation in the presence (absence) of the cosmological constant $\lambda = 16\Lambda/9M_p^2$. Both figures show that due to the contribution to the potential energy, $(-r^4/a^2)$, coming from the KR antisymmetric tensor sector of the theory, the potential develops an "abyss" in the small-a region regardless

of the presence or absence of the cosmological constant term. Since the WD equation implies that the total energy of the gravity-matter system is zero, $E = 0$, the emergence of the abyss in the small-a region of the potential readily reveals the fact that the universe wave function $\Psi[a]$ should be a highly oscillating function of a there. And this small-a behavior of the universe wave function, namely the enormous oscillation for small scale factor a appears to signal the existence of "quantum wormhole" as well as other types of spacetime fluctuations in the small-a region of the superspace and hence seems consistent with the existence of classical wormhole solution in this Einstein-KR antisymmetric tensor theory as we have seen in the earlier sections. Obviously, the most straightforward way of confirming the possible existence of "quantum wormholes" is to solve the WD equation given above for the universe wave functiion. Unfortunately, exact, analytic solutions to the WD equation in the presence of the cosmological constant are not available (exact solutions to the WD equation even for de Sitter spacetime pure gravity are not available either [14,15]). In the absence of the cosmological constant, $\lambda = 0$, however, an exact, analytic solution to the WD equation is available. There, of course, is a well-known issue of initial or boundary condition for the universe wave functioin. The WD equation, which plays the role of Schrődinger-type equation for the universe state, is a second order hyperbolic functional differential equation describing the evolution of the universe wave functioin in superspace. Thus the WD equation, in general, has a large number of solutions and in order to have any predictive power, one needs initial or boundary conditions to pick out just one solution by, for instance, giving the value of universe wave function at the boundary of the superspace on which it is defined. Thus far, a number of different proposals for the law of initial or boundary conditioins have been put forward. And among them, "no-boundary proposal" of Hartle and Hawking (HH) [14] and "tunnelling boundary condition" due to Vilenkin [15] are the ones which are the most comprehensive and the most extensively studied. If stated briefly, the no-boundary proposal by HH [14] is based on the philosophy that the quantum state of the universe is the closed cosmology's version of "ground state" or "state of minimal excitation" and the wave function of this ground state is given by an Euclidean sum-over-histories. Next, Vilenkin's tunnelling

boundary condition [15] can be best stated in "outgoing modes" formulation which governs the behavior of the solutions to the WD equation at boundaries of the superspace. Namely, according to this proposal, at "singular boundaries" (such as the region of zero 3-metric and infinite 3-curvature ($\sqrt{h} \to 0$) of superspace), the universe wave function should consist solely of outgoing modes carrying flux out of superspace. In practice, these two proposals for the law of initial or boundary conditions essentially aim at giving particular boundary conditions on the universe wave function at singular boundaries ($\sqrt{h} \to 0$) of the superspace which, presumably, are the points where the universe (or the universe wave function) has started. These boundary conditions, then, determine the behaviors of the universe wave function like how the universe nucleated (from "nothing") and then following which line it has subsequently evolved to the present one. Indeed, in the minisuperspace model (where the "singular" boundary corresponds to the point $a \to 0$) and within the context of the semiclassical approximation, these two proposals have been successfully applied to and tested for simple systems such as de Sitter spacetime pure gravity or a scalar field theory coupled minimally to gravity concretely demonstrating the ways how the universe nucleates and then subsequently evolves. Therefore, in view of this, applying these boundary conditions on the universe wave function (particular Vilenkin's tunnelling boundary condition) to the present case is, in many respects, irrelevant or, at least, awkward since we are supposed to determine the universe wave function in the small-a region, namely on the boundary itself of the (mini)superspace. Namely, for the case at hand, i.e., in the Einstein-KR antisymmetric tensor theory in the absence of the cosmological constant, the shape of the potential (given in Fig.4) as it appears in the WD equation "traps" the universe (namely the value of the scale factor, a) within a small-a region and we would like to determine the state of the universe in this region. Consequently, questions like how the universe nucleates or how it subsequently evolves are irrelevant. (Of course, HH's no-boundary proposal might still be relevant to be considered even for the present case since its formulation is based on the philosophy with wide applicability to general situations.) Therefore in the following, we present an exact, analytic solution to the WD equation in the absence of the cosmological constant that is

obtained by directly integrating the WD equation and is not constructed from any of these boundary conditions. Further, since this exact solution is a mathematical one, later we shall impose some conditions on the parameters involved in the solution in order for the resulting universe wave function to have physically relevant interpretations. Now consider the WD equation as given earlier but in this time in the absence of the cosmological constant. And in the following discussion, we shall set the constant n appearing in the $SO(4)$-symmetric ansätz for the KR field strength $h(t) = n/a^3(t)$ to be $n = (\sqrt{24}\pi\sigma^2)^{-1}$ so that the parameter $r^2 = \sqrt{24}\pi\sigma^2 n$ becomes unity. In fact, this rescaling amounts to taking natural unit in which $M_p^2 = 1$. The WD equation, then, takes the form

$$\Big[\frac{\partial^2}{\partial a^2} + \frac{p}{a}\frac{\partial}{\partial a} - a^2 + \frac{1}{a^2}\Big]\Psi[a] = 0. \tag{89}$$

An exact solution to this ordinary, second-order differential equation is given by

$$\begin{aligned}\Psi[a] &= Ca^{(\frac{1-p}{2})}Z_\nu(\frac{i}{2}a^2) \\ &= a^{(\frac{1-p}{2})}\Big[AJ_\nu(\frac{i}{2}a^2) + BN_\nu(\frac{i}{2}a^2)\Big]\end{aligned}$$

where $Z_\nu(z)$ is the Bessel function satisfying the Bessel equation and hence is generally given by the linear combination of the Bessel function of the 1st kind $J_\nu(z)$ which is regular for $z \to 0$ and the Bessel function of the 2nd kind (i.e., Neumann function) $N_\nu(z)$ which is regular for $z \to \infty$. And here the order of the Bessel function is given by $\nu = \frac{1}{4}\sqrt{(p+1)(p-3)}$ with p being the suffix indicating the ambiguity in "operator-ordering". A, B and C are arbitrary constant coefficients yet. Note that the structure of the WD equation above indicates that we are dealing with an one-dimensional Schrődinger-type equation with the total energy $E = 0$ and the potential given in Fig.4. Thus we expect that the physical solution, i.e., the universe wave function $\Psi[a]$ possesses a highly oscillating behavior for $a \to 0$ whereas a rapidly damping behavior for large a. The exact solution to the WD equation given above is yet just a mathematical solution. Now, we would like to turn it into a physical universe wave function by imposing physical conditions, namely by demanding that it satisfy appropriate asymptotic behaviors stated above. Fortunately, the exact solution

above involves an undetermined parameter p which is the index representing the operator-ordering ambiguity. Since this parameter p controls the behavior of the solution, namely the Bessel function, we shall be able to obtain a physical solution by fixing its value in such a way that with certain values of p the exact solution takes on expected asymptotic behavior stated above. Therefore, to this end, we carefully consider the asymptotic behaviors of the Bessel function. First for $z \to 0$, $Z_\nu(z) = J_\nu(z) \sim z^\nu/2^\nu \nu!$. Thus for $a \to 0$

$$Z_\nu(\frac{i}{2}a^2) \sim \frac{1}{2^\nu \nu!}(\frac{i}{2}a^2)^\nu \sim a^{2\nu}. \tag{90}$$

Now, in order for the universe wave function $\Psi[a] \sim a^{(\frac{1-p}{2})} a^{2\nu}$ with $\nu = \frac{1}{4}\sqrt{(p+1)(p-3)}$ in the region of small-a to have enormously oscillating behavior, the order ν should be imaginary, $\nu = i|\nu|$ which amounts to choosing $-1 < p < 3$. Consequently, the behavior of the universe wave function for $a \to 0$ is given by

$$\Psi[a] \sim a^{(\frac{1-p}{2})} \exp\left[i2|\nu| \ln a\right] \tag{91}$$

where $|\nu| = \frac{1}{4}\sqrt{|(p+1)(p-3)|}$ with $-1 < p < 3$. Apparently, this universe wave function possesses highly oscillatory behavior for $a \to 0$ and hence possibly represents a wave function of small scale spacetime fluctuations including wormholes. Note also that $\Psi[a] = 0$ at $a = 0$, namely it becomes regular for $-1 < p < 1$. Next, for $z \to$ large, $J_\nu(z) \sim \sqrt{\frac{2}{\pi z}} cos(z - \frac{\nu\pi}{2} - \frac{\pi}{4})$ and $N_\nu(z) \sim \sqrt{\frac{2}{\pi z}} sin(z - \frac{\nu\pi}{2} - \frac{\pi}{4})$, thus $Z_\nu(z) \sim \frac{1}{\sqrt{z}} e^{\pm iz}$. Therefore, for $a \to$ large

$$Z_\nu(\frac{i}{2}a^2) \sim \frac{1}{a} e^{\pm\frac{1}{2}a^2} \tag{92}$$

and hence the universe wave function behaves in the region of large-a as

$$\Psi[a] \sim a^{-(\frac{1+p}{2})} e^{-\frac{1}{2}a^2} \tag{93}$$

where we choose the minus sign in the exponent since it is the physically relevant one. Namely, the universe wave function possesses rapidly damping behavior for $a \to$ large and this is exactly what we expected. Finally the physically relevant solution to the WD equation in the Einstein-KR antisymmetric tensor theory (in the absence of the cosmological

constant), namely the universe wave function possibly of the quantum wormholes is given by

$$\Psi[a] = Ca^{(\frac{1-p}{2})} Z_{i|\nu|}(\frac{i}{2}a^2) = Ca^{(\frac{1-p}{2})} J_{i|\nu|}(\frac{i}{2}a^2) \tag{94}$$

where $|\nu| = \frac{1}{4}\sqrt{|(p+1)(p-3)|}$ with $-1 < p < 1$ and we dropped the Neumann function term demanding that the universe wave function remain finite for $a \to 0$. We believe that the solution to the WD equation given in eq.(94) would represent quantum wormhole spacetimes. Now, we would like to fortify this belief of ours in an unambiguous manner. As had been advocated by Hawking and Page [12], in order for a solution to a WD equation to represent quantum wormholes, it should obey certain boundary conditions. And the appropriate boundary conditions for wormhole wave functions seem to be that they are damped, say, exponentially for large 3-geometries ($\sqrt{h} \to \infty$) and are regular in some suitable way when the 3-geometry collapses to zero ($\sqrt{h} \to 0$). Particularly in the context of FRW minisuperspace model, large 3-geometries correspond to $a \to$ large limit and the 3-geometry collapsing to zero corresponds to $a \to 0$ limit. And the damping behavior of the universe wave function at large -a indicates that there are no gravitational excitations asymptotically and hence it represents asymptotically Euclidean spacetime while its regularity at $a = 0$ indicates that it is nonsingular. Therefore the solution to a WD equation obeying these boundary conditions must correspond to wormholes that connect two asymptotically Euclidean regions. Now we turn to our solution to the WD equation in Einstein-KR antisymmetric tensor theory given in eq.(94) and see if it indeed obeys these boundary conditions. Although we took the natural unit in which $M_p^2 = 1$ and rescaled such that $r^2 = \sqrt{24}\pi\sigma^2 n = 4\sqrt{6}n/3M_p^2$ takes the value of unity, we recover this length parameter for the moment. Firstly for $a \to 0$ or more concretely for $0 < a < r$, the universe wave function behaves like $\Psi \sim a^{(\frac{1-p}{2})} \exp[i2|\nu| \ln a]$ thus it oscillates infinitely and hence would correspond to initial or final spacetime singularities. In addition for $-1 < p < 1$, this solution is regular, i.e., $\Psi[a] = 0$ at $a = 0$. Secondly for $a \to$ large or more concretely for $a > r$, the universe wave function behaves as $\Psi \sim a^{-(\frac{1+p}{2})} e^{-a^2/2}$ thus it damps rapidly enough and thus represents asymptotically Euclidean regions. Namely

our solution does satisfy the boundary conditions for wormhole wave functions. Besides, the lower bound $a = r$ of the oscillating solution on the radius a of S^3 and the existence of the conserved axion current flux through the S^3 of the solution, i.e.,

$$\int_{S^3} H = \int_{S^3} \frac{h(t)}{f_a}\epsilon = \frac{2\pi^2 n}{f_a} \tag{95}$$

indicates that indeed our solutioin describes a wormhole connecting two asymptotically Euclidean regions. Finally, since our solution oscillates infinitely near $a = 0$, it would be expressible as an infinite sum of a discrete family of solutions to the WD equation that are well-behaved both at zero radius (i.e., "regularity") and at infinity (i.e., "damping"). And this completes the study of the solution to the WD equation in Einstein-KR antisymmetric tensor theory in the absence of the cosmological constant. It is interesting to note that our knowledge on the nature of the universe wave function in the absence of the cosmological constant developed thus far may, in turn, enable us to construct the solution to the WD equation in the presence of the cosmological constant at least approximately yet quite systematically. Thus in what follows, we shall turn to this problem. Now, we go back and consider the WD equation in Einstein-KR antisymmeric tensor theory in the presence of the cosmological constant

$$\frac{1}{2}\Big[-\frac{1}{a^p}\frac{\partial}{\partial a}(a^p\frac{\partial}{\partial a}) + (a^2 - \lambda a^4 - \frac{1}{a^2})\Big]\Psi[a] = 0. \tag{96}$$

With the potential energy, $\tilde{U}(a) = (a^2 - \frac{1}{a^2})$, as it appears in the WD equation in the absence of the cosmological constant, we now know that the solution to the Wheeler-DeWitt (WD) equation represents a quantum wormhole and particularly near $a = 0$, it oscillates infinitely and hence corresponds to large spacetime fluctuations. Therefore this observation plus the shape of the full potential energy, $\tilde{U}(a) = (a^2 - \lambda a^4 - \frac{1}{a^2})$ in the presence of the cosmological constant as was depicted in Fig.3 suggest that the solution to the WD equation above would describe the state of the universe that undergoes "large spacetime fluctuatioins for very small-a"$\rightarrow$ "spontaneous nucleation (quantum tunnelling) of the universe from nothing in a de Sitter geometry" $\rightarrow$ "subsequent, mainly classical evolution of the

universe for large-a". Namely, if we are willing to accept $(\partial/\partial a)$ as the timelike killing field in the (mini)superspace, the Einstein-antisymmetric tensor theory in the presence of the cosmological constant appears to serve as a simple yet interesting model which provides a comprehensive overall picture of entire universe's history from the deep quantum domain all the way to the essentially classical domain. Then coming back to a practical problem, now we wish to construct the approximate solutions to the WD equation in eq.(96). Clearly, the behavior of the solution for very small-a will be determined by the wormhole wave function obtained in the present work while the behavior for intermediate-to-large-a regions will be governed by the de Sitter space universe wave function. And as mentioned earlier, the de Sitter space universe wave functions has been constructed and extensively studied in the context of semiclassical approximation with the choice of both HH's no-boundary proposal [14] and Vilenkin's tunnelling boundary condition [15]. Thus to work out this idea, we consider the WD equation in two regions of interest in the minisuperspace. Firstly for very small-a region, the WD equation reduces to eq.(89) and the exact solution to this equation representing particularly the large spacetime fluctuations near $a = 0$ is given by

$$\begin{aligned} \Psi_I[a] &= a^{(\frac{1-p}{2})} J_{i|\nu|}(\frac{i}{2}a^2) \\ &\to a^{(\frac{1-p}{2})} \exp{[i2|\nu|\ln a]} \qquad (\text{for} \quad a \to 0) \\ &\to a^{-(\frac{1+p}{2})} e^{\pm\frac{1}{2}a^2} \qquad (\text{for} \quad a \to \text{large}) \end{aligned} \tag{97}$$

where $|\nu| = \frac{1}{4}\sqrt{|(p+1)(p-3)|}$ with $-1 < p < 1$. Secondly for intermediate-to-large-a regions, the WD equation reduces to

$$\Big[\frac{\partial^2}{\partial a^2} + \frac{p}{a}\frac{\partial}{\partial a} - a^2 + \lambda a^4\Big]\Psi[a] = 0.$$

As mentioned, the semiclassical approximation to the solutions of this de Sitter space WD equation has been thoroughly studied. First HH's no-boundary wave function [14] is given by

$$\Psi^{HH}_{II}[a] = a^{-(\frac{1+p}{2})} \exp\Big[\frac{1}{3\lambda}\{1 - (1-\lambda a^2)^{3/2}\}\Big] \qquad (\text{for} \quad \lambda a^2 < 1)$$

$$
\begin{aligned}
&\rightarrow a^{-(\frac{1+p}{2})} e^{\frac{1}{2}a^2} \quad (\lambda a^2 << 1), \qquad (98)\\
&= a^{-(\frac{1+p}{2})} \exp[\frac{1}{3\lambda}] 2cos\Big[\frac{(\lambda a^2 - 1)^{3/2}}{3\lambda} - \frac{\pi}{4}\Big] \qquad (\text{for} \quad \lambda a^2 > 1)\\
&\rightarrow a^{-(\frac{1+p}{2})} \Big[e^{i\frac{\sqrt{\lambda}}{3}a^3} + e^{-i\frac{\sqrt{\lambda}}{3}a^3}\Big] \quad (\lambda a^2 >> 1).
\end{aligned}
$$

This HH's no-boundary wave function consists of both "ingoing"(contracting) and "outgoing"(reexpanding) modes in the classical-allowed region ($\lambda a^2 > 1$) which, then, decreases exponentially as it moves towards smaller values of a in the classically-forbidden region ($\lambda a^2 < 1$). Next, Vilenkin's tunnelling wave function [15] is given by

$$
\begin{aligned}
\Psi^T_{II}[a] &= a^{-(\frac{1+p}{2})} (1 - \lambda a^2)^{-1/4} \exp\Big[-\frac{1}{3\lambda}\{1 - (1 - \lambda a^2)^{3/2}\}\Big] \qquad (\text{for} \quad \lambda a^2 < 1)\\
&\rightarrow a^{-(\frac{1+p}{2})} e^{-\frac{1}{2}a^2} \quad (\lambda a^2 << 1), \qquad (99)\\
&= a^{-(\frac{1+p}{2})} e^{i\frac{\pi}{4}} (\lambda a^2 - 1)^{-1/4} \exp\Big[-\frac{1}{3\lambda}\{1 + i(\lambda a^2 - 1)^{3/2}\}\Big] \qquad (\text{for} \quad \lambda a^2 > 1)
\end{aligned}
$$

This Vilenkin's tunnelling wave function exponentially decreases as it moves from small toward larger values of a (i.e., emerges out of the potential barrier via "quantum tunnelling") and then upon escaping the barrier, it consists solely of "outgoing" (expanding) mode in the classically-allowed region ($\lambda a^2 > 1$). Now we are ready to write down approximate solutions to the WD equation in the presence of the cosmological constant by putting these pieces altogether. To do so let us denote the smaller and larger roots of the equation

$$
\tilde{U}(a) = a^2 - \lambda a^4 - \frac{1}{a^2} = 0
$$

by r_- and $r+$ respectively. Note that this equation has positive roots provided $\Lambda < (\frac{3}{8})^2 M_p^4$ or $\lambda < 1/4$ and the two roots are given by $r^4_\pm = \frac{1}{2\lambda}[1 \pm \sqrt{1 - 4\lambda}]$.

(1) With the choice of HH's no-boundary wave function :

$$
\begin{aligned}
\Psi[a] &= \Psi_I[a] \qquad (\text{region } I : 0 < a < r_+),\\
&= \Psi^{HH}_{II}[a] \qquad (\text{region } II : r_- < a < \infty).
\end{aligned}
$$

(2) With the choice of Vilenkin's tunnelling wave function :

$$\Psi[a] = \Psi_I[a] \qquad\qquad (\text{region } I : 0 < a < r_+),$$
$$= \Psi^T_{II}[a] \qquad\qquad (\text{region } II : r_- < a < \infty).$$

The two kinds of universe wave functions corresponding to the two different choices of the boundary conditions are plotted in Fig 5. and 6 respectively.

V. Discussions

Now we summarize the motivation and the results of the present work.

We revisited, in this work, the Einstein-KR antisymmetric tensor theory considered first by Giddings and Strominger [4] which is a classic system known to admit classical, Euclidean wormhole instanton solution. Although the classical wormhole instanton as a solution to the classical field equations and much of its effects on low energy physics have been studied extensively in the literature, some of important aspects of the classical wormhole physics such as the existence and the physical implications of fermion zero modes in the background of classical axionic wormhole spacetime has not been addressed. Moreover, since this Einstein-KR antisymmetric tensor system admits classical wormhole solutions, one may wonder if there is any systematic way of exploring the existence and the physics of "quantum" wormholes in the same theory. The present work attempted to deal with these kinds of yet unquestioned issues. And firstly, in order to investigate the existence of the fermion zero modes and their physical implications, we followed the formulation taken by Hosoya and Ogura [5] in their study of classical wormhole instantons in Einstein-Yang-Mills theory. And to do so, we needed to introduce the fermion-KR antisymmetric tensor field interactions possessing, of course, the general covariance and the local gauge-invariance. And the result was that regardless of the gauge choices associated with the time reparametrization invariance, i.e., $N(\tau) = 1$ or $a(\tau)$, there are two normalizable fermion zero modes. As we mentioned in the text, the existence of fermion zero modes would affect wormhole interactions. Namely, the fermion zero modes, upon integration, would yield a long-range confining interaction between the axionic wormholes. Next, the fermion zero modes, i.e., the solutions to the massless Dirac equation, are symmetric with respect to the chirality flip. And this may sig-

nal that the axionic wormhole instantons would not induce the chirality-changing fermion propagation unlike the typical instantons in non-abelian gauge theories. Secondly, in order to explore the quantum wormholes in this system systematically, we worked in the context of canonical quantum cosmology and followed Hawking and Page [12] to define the quantum wormhole as a state or an excitation represented by a solution to the Wheeler-DeWitt equation satisfying a certain wormhole boundary condition. Particularly, in the minisuperspace quantum cosmology model possessing SO(4)-symmetry, an exact, analytic solution to the Wheeler-DeWitt equation satisfying the appropriate wormhole boundary condition was found in the absence of the cosmological constant. Thus we confirmed our expectation that the Einstein-KR antisymmetric system admits quantum wormholes as well as classical wormholes. Further, we pointed out that the minisuperspace quantum cosmology model based on this Einstein-KR antisymmetric tensor theory in the presence of the cosmological constant may serve as an simple yet interesting system displaying an overall picture of entire universe's history from the deep quantum domain all the way to the classical domain.

As we have stressed in the text, the essential point that allowed us to explore, in a concrete manner, the quantum wormhole in the context of the minisuperspace quantum cosmology model was the following observation. In their original work, Giddings and Strominger [4] looked for an economical way of solving the coupled Einstein-KR antisymmetric tensor field equations. They found out that the classical Euler-Lagrange's equation of motion $d^*H = 0$ and the Bianchi identity $dH = 0$ can be simultaneously satisfied if one takes the SO(4)-symmetric ansätz for the KR antisymmetric tensor field strength as $H_{\mu\nu\lambda} = \frac{n}{f_a^2 a^3}\epsilon_{\mu\nu\lambda}$ which, in turn, reduces the Einstein equation to that of the scale factor $a(\tau)$ alone. However, we realized in this work that even without imposing the on-shell condition (i.e., the classical field equation), one can "derive" $H_{\mu\nu\lambda} = \frac{const.}{a^3}\epsilon_{\mu\nu\lambda}$ just from the definition $H = dB$ and the Bianchi identity $dH = 0$. Therefore this SO(4)-symmetric ansätz for the KR antisymmetric tensor field strength $H_{\mu\nu\lambda}$ remains valid even off-shell as well as on-shell and hence can be used in the quantum treatment of the Einstein-KR antisymmetric tensor field system. Consequently, the Wheeler-DeWitt equation in the context of the canonical quantum cosmology

becomes a Schrődinger-type equation of the minisuperspace variable a (the scale factor) alone and can be solved exactly particularly in the absence of the cosmological constant. Finally, our study of quantum wormholes in this work appears to demonstrate that, after all, the Einstein-KR antisymmetric tensor theory is a simple (although it is a truncated system of a more involved, fundamental string theory) but fruitful system which serve as an arena in which we can envisage quite a few exciting aspects of quantum gravitational phenomena.

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References

[1] E. Baum, Phys. Lett. **B133**, 185 (1983) ; S. W. Hawking, Phys. Lett. **B134**, 403 (1984).

[2] S. Giddings and A. Strominger, Nucl. Phys. **B307**, 854 (1988).

[3] S. Coleman, Nucl. Phys. **B307**, 864 (1988) ; Nucl. Phys. **B310**, 643 (1988) ; I. Klebanov, L. Susskind, and T. Banks, Nucl. Phys. **B317**, 665 (1989).

[4] S. Giddings and A. Strominger, Nucl. Phys. **B306**, 890 (1988) ; S. -J. Rey, Nucl. Phys. **B319**, 765 (1989).

[5] A. Hosoya and W. Ogura, Phys. Lett. **B22**, 117 (1989) ; S. -J. Rey, Nucl. Phys. **B336**, 146 (1990).

[6] K. Lee, Phys. Rev. Lett. **61**, 263 (1988).

[7] B. Grinstein and M. B. Wise, Phys. Lett. **B212**, 407 (1988) ; B. Grinstein, Nucl. Phys. **B321**, 439 (1989).

[8] W. Fischler and L. Susskind, Phys. Lett. **B217**, 48 (1989).

[9] S. W. Hawking, Commun. Math. Phys. **43**, 199 (1975).

[10] S. W. Hawking, Phys. Rev. **D14**, 2460 (1976) ; For a recent review, see R. Wald, presented at 1991 Erice lectures (unpublished).

[11] G. W. Gibbons, S. W. Hawking, and M. J. Perry, Nucl. Phys. **B138**, 141 (1978).

[12] S. W. Hawking and D. N. Page, Phys. Rev. **D42**, 2655 (1990).

[13] C. W. Misner, K. S. Thorne, and J. A. Wheeler, *Gravitation* (W. H. Freeman, San Francisco, 1973).

[14] J. B. Hartle and S. W. Hawking, Phys. Rev. **D28**, 2960 (1983) ; S. W. Hawking, Nucl. Phys. **B239**, 257 (1984).

[15] A. Vilenkin, Phys. Lett. **B117**, 25 (1982) ; Phys. Rev. **D30**, 509 (1984) ; Nucl. Phys. **B252**, 141 (1985) ; Phys. Rev. **D33**, 3560 (1986) ; Phys. Rev. **D37**, 888 (1988).

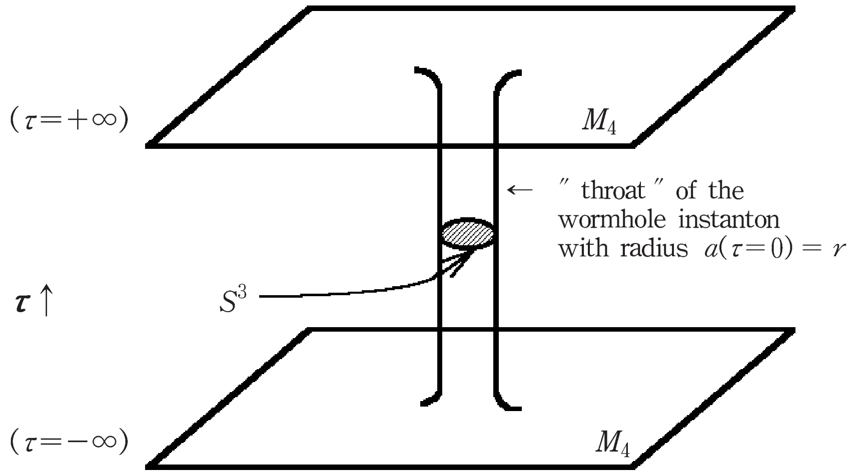

Fig.1 A wormhole with two asymptotically-Euclidean regions.
Two-dim. are suppressed: the cross section of the throat really represents a 3-sphere, S^3.

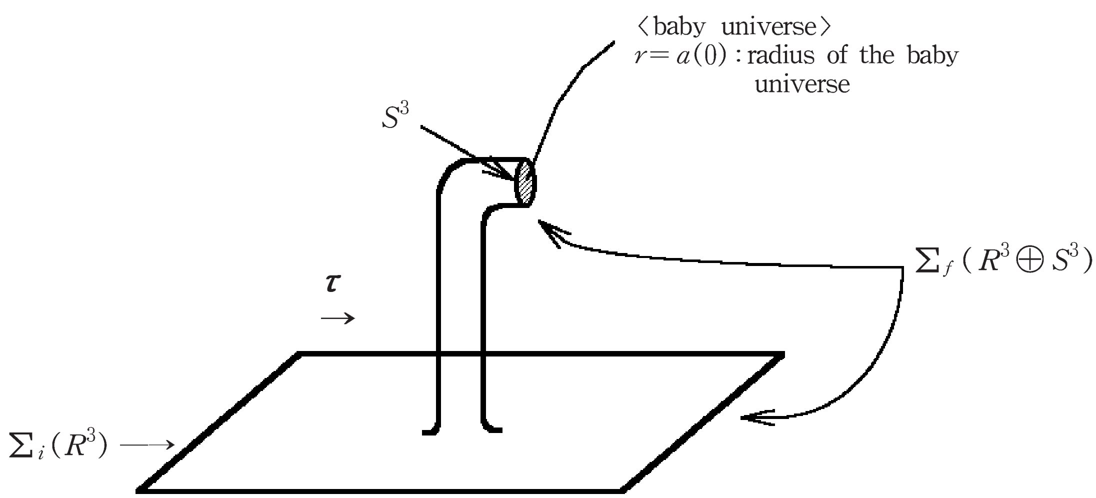

Fig.2 A instanton representing topology change from R^3 to $R^3 \oplus S^3$ and vice versa.

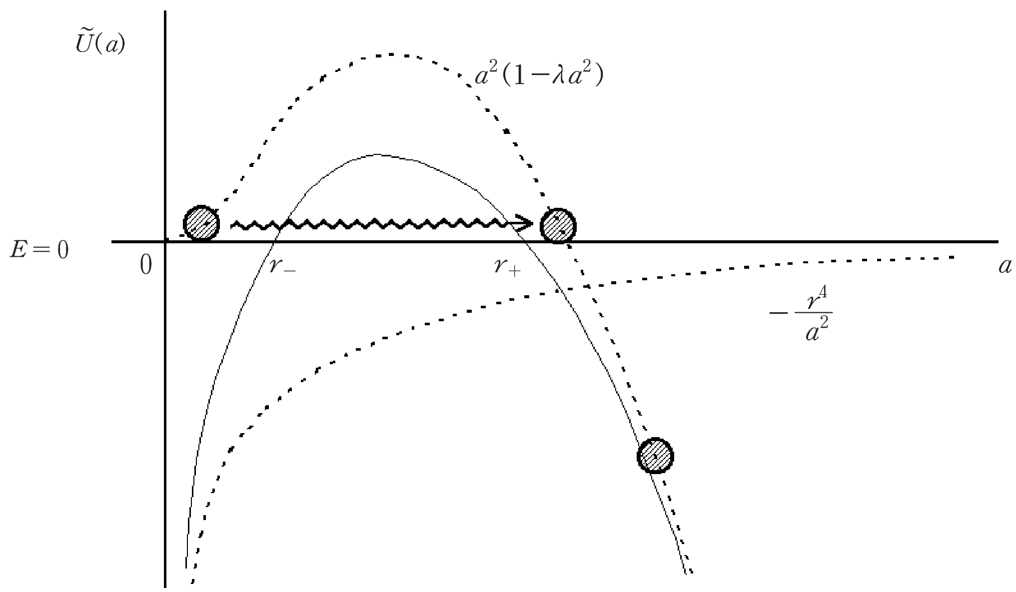

Fig.3 The potential energy as it appears in the WD equation in the presence of the cosmological constant.

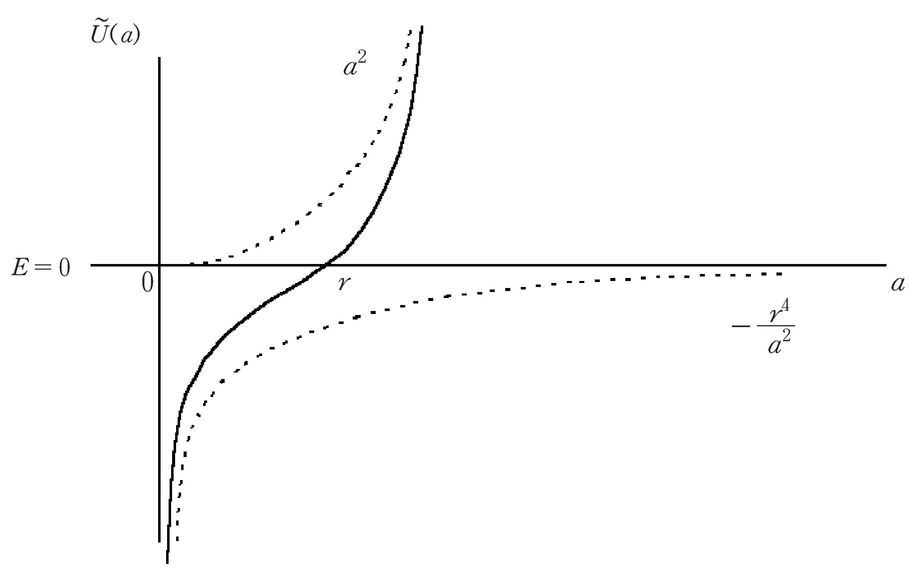

Fig.4 The plot of potential energy as it appears in the WD equation in the absence of the cosmological constant.

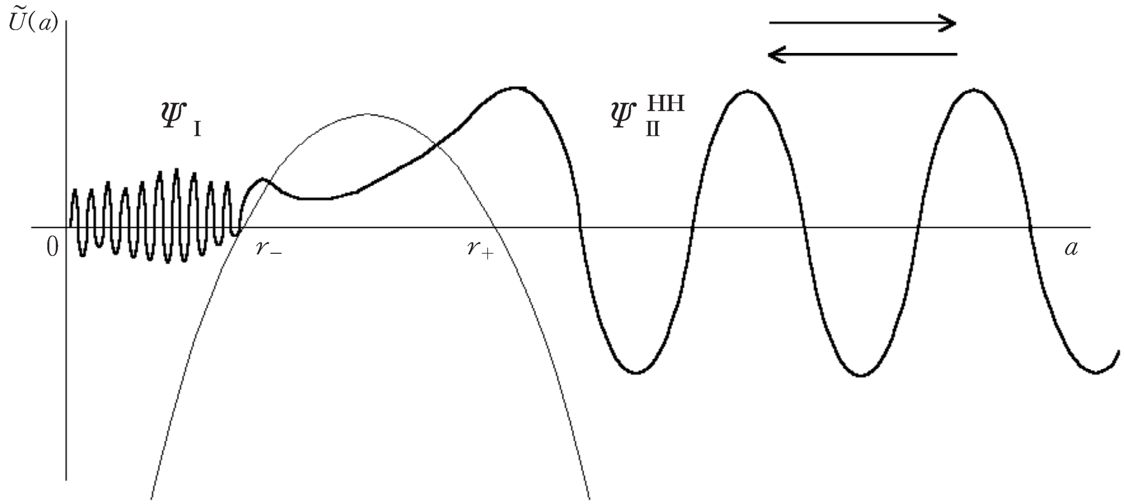

Fig.5 A regular solution to the WD equation in the presence of the cosmological constant employing HH's no-boundary proposal.

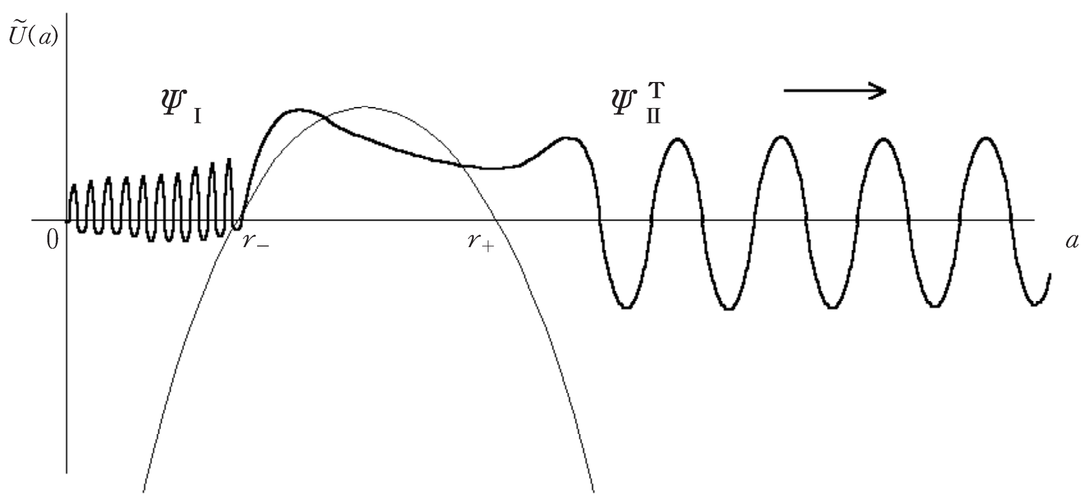

Fig.6 A regular solution to the WD equation in the presence of the cosmological constant employing Vilenkin's tunnelling boundary condition.